QUATERNION DEVELOPMENTS WITH APPLICATIONS*

BY

JAMES BYRNIE SHAW

INTRODUCTION.

This paper is an extension of the quaternion algebra along lines analogous to those of a preceding paper.† Certain functions of one or more quaternions are studied in detail, all of them independent of the orientation of the unit axes, $1, i, j, k$. The notation is functional and seems preferable to the use of brackets such as those used by HAMILTON,‡ who, indeed, called his notation temporary. We seem to have here a natural method for treating several functions that occur frequently in analysis. For example, if we represent quaternion numbers by vectors in four-dimensional space, as was done by STRINGHAM§ and HATHAWAY,|| these functions are expressions for geometrical relations which are independent of the particular system of rectangular axes used. These expressions are useful in electrodynamics and in relativity problems. They include the Hamiltonian $[ab]$, (abc), $[abc]$, $(abcd)$, and the equivalent forms used by JOLY,¶ and SCHRUTKA VON RECHTENSTAMM.**

The development of these formulæ enable us also to formulate certain differential operators of frequent occurrence, which, from their nature, produce expressions, each invariant under orthogonal transformation in the sense that after the transformation has been performed on the various quaternions of which it is a function, the result is the transform of the initial function (§ 8). Certain of these forms are also invariant under any linear homogeneous transformation. Similar forms were used by MACMAHON†† in the study of the ternary and quaternary invariants and covariants under orthogonal transformations. The study of differential parameters and their syzygies is made

* Presented to the Society (Chicago), April 8, 1910, and April 28, 1911.

† American Journal of Mathematics, vol. 19 (1897), pp. 193–216.

‡ *Elements of Quaternions*, 2d ed., vol. I, page 563.

§ These Transactions, vol. 2 (1901), pp. 183–214.

|| These Transactions, vol. 3 (1902), pp. 46–59.

¶ *Manual of Quaternions*, page 264.

** Wiener Sitzungsberichte, vol. 115 (1906), Abt. II*a*, pp. 739–775.

†† Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1904), pp. 210–229.

simpler by the use of these forms, and the symbolic invariants of MASCHKE * appear as a natural part of the development. Two differential operators are introduced which are the extensions of the quaternion ∇ to space of four dimensions and to curvilinear three-dimensional space.

Up to the present time, the greater part of the development of the quaternion algebra has been in terms of scalars and vectors rather than in terms of quaternions proper. This often introduces prolixity in the development. It is hoped that the present paper will present some fruitful extensions along the more general line. In the expansion of the forms appearing here it will be evident that they are of the nature of determinants and pfaffians. That other forms might be studied is of course self-evident, but these are the ones of most frequent occurrence in the functions of geometry and physics. However the developments below are on the formal side and valid irrespective of the interpretation of the quaternions.

PART I. FUNDAMENTAL FORMULÆ.

1. Throughout the paper italic letters will be used to represent quaternions, which may occasionally reduce to scalars or vectors. The conjugate of a quaternion q will usually be denoted by q', but the conjugate of an expression, as qrs, will be denoted by $\mathrm{K}\cdot qrs$.

2. We notice that the quaternions uv' and $v'u$ are important, particularly on account of the relations shown in the formulæ

$$uv' = \mathrm{S}\cdot uv' + \mathrm{V}u\mathrm{S}v - \mathrm{V}v\mathrm{S}u - \mathrm{V}\mathrm{V}u\mathrm{V}v,$$

which resolves uv' into three parts whose products have vanishing scalars; and if l and m are scalars,

$$uv'(lu + mv) = u(2l\mathrm{S}uv' + m\mathrm{T}^2 v) - lv\mathrm{T}^2 u = (lu + mv)v'u,$$

which shows that any quaternion of the form $lu + mv$ is converted into another of the same form by left-hand multiplication by uv' or by right-hand multiplication by $v'u$. Also uv' and $v'u$ differ only in their axes, having the same tensors and angles. The quaternions uv' and vu' however differ only in having opposite angles. If $r = lu + mv$, we have $uv'r - rv'u = 0$.

3. We define first the two functions

$$\mathrm{I}\cdot uv = \tfrac{1}{2}(uv' + vu'), \qquad \mathrm{A}_2\cdot uv = \tfrac{1}{2}(uv' - vu').$$

The first of the two functions is obviously a scalar, since it is the sum of a quaternion and its conjugate, and the second is a vector. Indeed,

$$\mathrm{I}\cdot uv = \mathrm{S}\cdot uv' = \mathrm{T}^2 v\cdot \mathrm{S}\cdot\frac{u}{v}, \qquad \mathrm{A}_2\cdot uv = \mathrm{V}uv' = \mathrm{T}^2 v\cdot \mathrm{V}\cdot\frac{u}{v}.$$

* These Transactions, vol. 1 (1900), pp. 197–204; vol. 7 (1906), pp. 69–80.

From the definitions we have at once

$$\mathrm{I} \cdot uv = \mathrm{I} \cdot vu = \mathrm{I} \cdot u'v' = \mathrm{I} \cdot v'u', \qquad \mathrm{A}_2 \cdot uv = -\mathrm{A}_2 \cdot vu.$$

The latter expression is alternating and therefore vanishes if u is a scalar multiple of v. If we write $\rho = \mathrm{UVV}u\mathrm{V}v$, we find with no difficulty that

$$\mathrm{A}_2 \cdot u'v' = \rho \cdot \mathrm{A}_2\, uv \cdot \rho^{-1}.$$

Hence $\mathrm{A}_2 \cdot uv$ differs from $\mathrm{A}_2\, u'v'$ only in direction. By expanding the two sides of the equations, we find that

$$u^{-1}\,\mathrm{A}_2 \cdot uv \cdot u = \mathrm{A}_2 \cdot v'u', \qquad v^{-1} \cdot \mathrm{A}_2 \cdot uv \cdot v = -\mathrm{A}_2 \cdot u'v'.$$

The two forms used by Joly are, in this notation, as follows:

$$(uv) = v\mathrm{S}u - u\mathrm{S}v = \tfrac{1}{2}\,(\mathrm{A}_2 \cdot u'v' - \mathrm{A}_2 \cdot uv),$$

$$[uv] = \mathrm{VV}u\mathrm{V}v = \tfrac{1}{2}(-\mathrm{A}_2 \cdot u'v' - \mathrm{A}_2 \cdot uv).$$

The single form given by Hamilton, $[uv]$, is twice the corresponding one given by Joly, that is, is $2\,\mathrm{VV}u\mathrm{V}v$.

If $\mathrm{I} \cdot uv = 0$ we shall speak of u and v as orthogonal. The solution of the two equations which express that r is orthogonal to both u and v is

$$r = x\,(uv' - v'u) + y\,(uv'\,uv' - uv'\,v'u),$$

where x and y are arbitrary scalars; for, this value of r satisfies the two equations, and r can depend only upon two arbitrary parameters.

4. Passing now to three quaternions, we define the functions

$$\mathrm{A}_1 \cdot uvw = \tfrac{1}{2}\,(uv'\,w + wv'\,u), \qquad \mathrm{A}_3 \cdot uvw = \tfrac{1}{2}\,(uv'\,w - wv'\,u).$$

It is evident that $\mathrm{A}_1 \cdot uvw = \mathrm{A}_1 \cdot wvu$. Also

$$\mathrm{A}_1 \cdot uvw = \tfrac{1}{2}\,(vw'\,u + wv'\,u - vu'\,w - vw'\,u + uv'\,w + vu'\,w)$$

$$= u\mathrm{I} \cdot vw - v\mathrm{I} \cdot wu + w\mathrm{I} \cdot uv.$$

Again from the definition it is obvious that

$$\mathrm{A}_3 \cdot uvw = -\mathrm{A}_3 \cdot wvu.$$

Since we have identically $2u\mathrm{I} \cdot vw = 2\mathrm{I} \cdot vw \cdot u$, we have by expansion the identity

$$uv'\,w + uw'\,v = vw'\,u + wv'\,u,$$

from which by transposition, and by reapplying the result, we arrive at

$$uv'\,w - wv'\,u = vw'\,u - uw'\,v = wu'\,v - vu'\,w.$$

Therefore, returning to the form A_3, we have

$$A_3 \cdot uvw = A_3 \cdot vwu = A_3 \cdot wuv = -A_3 \cdot wvu = -A_3 \cdot uwv$$
$$= -A_3 \cdot vuw.$$

This function is therefore an alternating function of u, v, and w. It vanishes if they are linearly connected. Joly's form $[uvw]$ is in this notation $A_3 \cdot u'v'w'$. His form (uvw) is $SA_3 \cdot u'v'w'$. We may expand in the form

$$A_3 \cdot uvw = \tfrac{1}{6}\,[uv'w - uw'v + vw'u - vu'w + wu'v - wv'u]$$
$$= \tfrac{1}{3}\,[uA_2 \cdot v'w' + vA_2 \cdot w'u' + wA_2 \cdot u'v'].$$

This form shows that the function is of the nature of a determinant.* Again,

$$I \cdot uA_3 \cdot uvw = 0 = I \cdot vA_3 \cdot uvw = I \cdot wA_3 \cdot uvw.$$

For, if we expand $I \cdot uA_3 \cdot uvw$ we have

$$\tfrac{1}{4}\,[u'uv'w - u'wv'u + w'vu'u - u'vw'u]$$
$$= \tfrac{1}{4}\,[u'uI \cdot vw - u'I \cdot vw \cdot u] = 0.$$

A similar proof holds for v and w. Hence the solution of the three equations expressing the orthogonality of r to u, v, w, is $r = xA_3 \cdot uvw$, where x is an arbitrary scalar.

We see also from these equations and the expanded form of $A_1 \cdot uvw$ that we may resolve $uv'w$ into two orthogonal parts:

$$uv'w = A_1 \cdot uvw + A_3 \cdot uvw.$$

If $A_1 \cdot uvw = 0$, we have $uv'w + wv'u = 0$, whence

$$uv'wv'v + wv'uv'v = 0 = wv'uv'w + wv'wv'u.$$

Adding the two sides of this double equality we have

$$[wv'wv' + wv'(w'v + v'w) + vv'ww']\,u = 0.$$

This gives easily $wv'uI \cdot vw = 0$. Similarly we find that $uv'wI \cdot uw = 0$. Hence either we must have $I \cdot vw = 0 = I \cdot uw$ or else $uv'w = 0 = wv'u$. The latter cannot hold unless T^2u, or T^2v, or T^2w vanishes. Then if the quaternions are real we must have, when $A_1 \cdot uvw = 0$, the three equations $I \cdot uv = 0 = I \cdot uw = I \cdot vw$, and the three quaternions form an orthogonal system.

We find that†

$$TA_1 \cdot uvw = \sqrt{\,(Iuu I^2 vw + Ivv I^2 wu + Iww I^2 uv - 2Iuv Ivw Iwu)}$$
$$= TA_1 \cdot vwu = \cdots.$$

*Cf. JOLY: Transactions of the Royal Irish Academy, 1902, 32A, Part II, pp. 17–30.

†The period following I and A will be omitted when no confusion can result, just as it is after S and V in treatises on Quaternions.

Since $\mathrm{T}^2\, uv'\, w = 1$ when u, v, and w are unit quaternions, and

$$\mathrm{T}^2\, uv'\, w = \mathrm{T}^2\, \mathrm{A}_1 \cdot uvw + \mathrm{T}^2\, \mathrm{A}_3 \cdot uvw,$$

we might call $\mathrm{TA}_1 \cdot uvw$ the cosine of the triple u, v, w and $\mathrm{TA}_3 \cdot uvw$ the sine. The latter for geometric reasons could also be called the Staudtian. Moreover,

$$\mathrm{T}^2\, \mathrm{A}_3 \cdot uvw = \mathrm{I} \cdot \mathrm{A}_3 \cdot uvw\mathrm{A}_3 \cdot uvw = \begin{vmatrix} \mathrm{I}uu & \mathrm{I}uv & \mathrm{I}uw \\ \mathrm{I}vu & \mathrm{I}vv & \mathrm{I}vw \\ \mathrm{I}wu & \mathrm{I}wv & \mathrm{I}ww \end{vmatrix}.$$

5. If we write out each quaternion in terms of $1, i, j, k$ we find that

$$\mathrm{A}_3 \cdot uvw = \begin{vmatrix} 1 & i & j & k \\ u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \\ w_0 & w_1 & w_2 & w_3 \end{vmatrix}.$$

From this we have at once

$$\mathrm{I} \cdot t\mathrm{A}_3 \cdot uvw = \left| \begin{matrix} t_0 & u_1 & v_2 & w_3 \end{matrix} \right|.$$

Also from the properties of determinant multiplication we have the important formula

$$\mathrm{I} \cdot \mathrm{A}_3 \cdot abc\mathrm{A}_3 \cdot uvw = \begin{vmatrix} \mathrm{I}au & \mathrm{I}av & \mathrm{I}aw \\ \mathrm{I}bu & \mathrm{I}bv & \mathrm{I}bw \\ \mathrm{I}cu & \mathrm{I}cv & \mathrm{I}cw \end{vmatrix}.$$

We may also easily verify that

$$\mathrm{A}_3 \cdot bc\mathrm{A}_3 \cdot uvw = \begin{vmatrix} -u & -v & -w \\ \mathrm{I}bu & \mathrm{I}bv & \mathrm{I}bw \\ \mathrm{I}cu & \mathrm{I}cv & \mathrm{I}cw \end{vmatrix}.$$

Hence we have at once

$$\mathrm{I} \cdot a\mathrm{A}_3\, bc\mathrm{A}_3\, uvw = -\mathrm{IA}_3\, abc\mathrm{A}_3\, uvw.$$

6. Some useful formulæ are

$$\mathrm{A}_2 \cdot \mathrm{A}_2\, uv\mathrm{A}_2\, wx = -\mathrm{A}_2\, uw\mathrm{I}vx + \mathrm{A}_2\, ux\mathrm{I}vw + \mathrm{A}_2\, vw\mathrm{I}ux - \mathrm{A}_2\, vx\mathrm{I}uw,$$

$$\mathrm{A}_2 \cdot u\mathrm{A}_3\, vwx = -\mathrm{A}_2\, vw\mathrm{I}ux - \mathrm{A}_2\, wx\mathrm{I}uv - \mathrm{A}_2\, xv\mathrm{I}uw,$$

$$\mathrm{A}_3\, uvw \cdot u' = u\mathrm{A}_3\, u'\, v'\, w',$$

$$\mathrm{A}_3 \cdot \mathrm{A}_2\, uv\mathrm{A}_2\, vw\mathrm{A}_2\, wu = \mathrm{T}^2\, \mathrm{A}_3\, uvw.$$

7. For four quaternions we define the three functions

$$\mathrm{I}\cdot uvwx = \mathrm{I}u\mathrm{A}_1 vwx = \mathrm{I}uv\mathrm{I}wx - \mathrm{I}uw\mathrm{I}vx + \mathrm{I}ux\mathrm{I}vw$$
$$= \tfrac{1}{4}(uv'wx' + ux'wv' + vw'xu' + vu'xw') = \tfrac{1}{2}(\mathrm{S}\cdot uv'wx' + \mathrm{S}\cdot vw'xu').$$
$$\mathrm{A}_4\cdot uvwx = -\mathrm{I}u\mathrm{A}_3 vwx = \tfrac{1}{4}(uv'wx' - ux'wv' + xw'vu' - vw'xu')$$
$$= \tfrac{1}{2}(\mathrm{S}\cdot uv'wx' - \mathrm{S}\cdot vw'xu').$$
$$\mathrm{A}_2\cdot uvwx = \tfrac{1}{2}(uv'wx' - xw'vu') = \mathrm{V}\cdot uv'wx'.$$

Evidently in $\mathrm{I}uvwx$ we may permute the four quaternions cyclically, or may reverse the order. A_4 is an alternating function. In A_2 the order may be reversed by changing the sign. $\mathrm{I}uvwx$ is a pfaffian, for it is the square root of a skew symmetric determinant of even order; indeed we have by actually squaring the expanded form,

$$(\mathrm{I}uvwx)^2 = \begin{vmatrix} 0 & \mathrm{I}uv & \mathrm{I}uw & \mathrm{I}ux \\ -\mathrm{I}vu & 0 & \mathrm{I}vw & \mathrm{I}vx \\ -\mathrm{I}wu & -\mathrm{I}wv & 0 & \mathrm{I}wx \\ -\mathrm{I}xu & -\mathrm{I}xv & -\mathrm{I}xw & 0 \end{vmatrix}.$$

8. If now we define a function $\mathrm{A}_5\cdot uvwxy$ in the same manner as above it will be alternating and vanish, since any five quaternions are linearly connected. Indeed the vanishing of this expression gives us the important identity

$$u\mathrm{A}_4 vwxy - v\mathrm{A}_4 uwxy + w\mathrm{A}_4 uvxy - x\mathrm{A}_4 uvwy + y\mathrm{A}_4 uvwx = 0.$$

Operating by $\mathrm{I}\cdot z(\ )$, where z is any quaternion, we find the important identity

$$z\mathrm{A}_4 vwxy = -\mathrm{A}_3 wxy\mathrm{I}vz + \mathrm{A}_3 vxy\mathrm{I}wz - \mathrm{A}_3 vwy\mathrm{I}xz + \mathrm{A}_3 vwx\mathrm{I}yz.$$

We define further

$$\mathrm{A}_1\cdot uvwxy = \tfrac{1}{2}(uv'wx'y + yx'wv'u),$$
$$\mathrm{A}_3\cdot uvwxy = \tfrac{1}{2}(uv'wx'y - yx'wv'u).$$

Functions like these and the even numbered functions I, A_2, A_4 may be defined for any number of quaternions, and are useful in reductions.

As previously stated, all these forms are invariant or pseudo-invariant under the substitution of aqb for q, where $\mathrm{T}a = \mathrm{T}b = 1$. This is evident by mere substitution in the definitions. For example,

$$\mathrm{A}_2 uv = a'\,\mathrm{A}_2(aub)(avb)\cdot a, \qquad \mathrm{A}_1 uvw = a^{-1}\cdot\mathrm{A}_1(aub)(avb)(awb)\cdot b^{-1},$$
$$\mathrm{A}_3\cdot uvw = a'\cdot\mathrm{A}_3(aub)(avb)(awb)\cdot b',$$

while $\mathrm{I}\cdot uv$, $\mathrm{I}\cdot uvwx$, $\mathrm{A}_4\cdot uvwx$ are invariant.

The functions $\mathrm{A}_3\,uvw$ and $\mathrm{A}_4\,uvwx$ are invariant under any linear homogeneous substitution, in the sense that such a substitution merely multiplies the function by the determinant of the substitution.

PART II. DIFFERENTIAL OPERATORS.

9. Let us suppose that the coördinates of a point in space of four dimensions are w, x, y, z, corresponding to the real or scalar axis and the vector axes i, j, k. We then define the differentiating operator * D as follows:

$$\mathrm{D} = \frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

In physical applications it would be desirable to define this operator directly in physical terms, and in geometry in geometrical terms, but for the present formal developments this definition seems simpler. It is obvious that if the quaternion q becomes $q + dq$, where

$$dq = dw + idx + jdy + kdz,$$

then the operator

$$\mathrm{I} \cdot dq\mathrm{D} = dw\frac{\partial}{\partial w} + dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}$$

gives the differential change in Q, any function of the position q, due to the differential change dq. If Q is a scalar function of position and its levels given by $Q = c$, then the change in Q which is greatest will be normal to these levels and will be given by DQ itself. Let

$$Q = W + iX + jY + kZ.$$

Then

$$\mathrm{I} \cdot \mathrm{D}Q = \frac{\partial W}{\partial w} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}.$$

We might call this the *divergence* of the function Q. Its vanishing gives an equation well known in hydrodynamics, the equation of continuity. In this case we take $Q = \rho(1 + \sigma)$ where σ is the velocity, ρ the density, w the time.

If we combine D with itself we have

$$\mathrm{I} \cdot \mathrm{DD} = \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad \mathrm{I} \cdot \mathrm{DD}' = \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$

The first is the extension of the Laplacian to four dimensions and leads to what may be called four-dimensional potential functions. We have an example of the vanishing of these differential forms in the differential equation of wave motion, which if we set $w = at$ may be written

$$\mathrm{I} \cdot \mathrm{DD} \cdot u = 0, \qquad \text{or} \qquad \mathrm{I} \cdot \mathrm{DD}' \cdot u = 0,$$

according as the term containing the differential of the time is negative or positive.

* This is the operator *lor* of MINKOWSKI, Mathematische Annalen, vol. 68 (1909–10), pp. 472–525.

The first partial derivatives of a scalar function Q as to the coördinates, are the coördinates of the quaternions $1, i, j, k$ in $\mathrm{D}Q$. The second partial derivatives would be the coefficients of the dyads in $\mathrm{DID}Q$. In fact, the scalar invariants† of the linear quaternion differential operator $\mathrm{DI}\cdot\mathrm{D}(\,)$, namely M_1, M_2, M_3, M_4, lead not only to the Laplacian, but to other forms of operators, all of which are invariant under orthogonal transformations of the axes $1, i, j, k$.

We have

$$\mathrm{A}_2\,\mathrm{D}Q = i\left(\frac{\partial W}{\partial x}-\frac{\partial X}{\partial w}+\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)+j\left(\frac{\partial W}{\partial y}-\frac{\partial Y}{\partial w}+\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)$$
$$+k\left(\frac{\partial W}{\partial z}-\frac{\partial Z}{\partial w}+\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right),$$

$$\mathrm{A}_2\,\mathrm{D}'Q' = i\left(-\frac{\partial W}{\partial x}+\frac{\partial X}{\partial w}+\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)+j\left(-\frac{\partial W}{\partial y}+\frac{\partial Y}{\partial w}+\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)$$
$$+k\left(-\frac{\partial W}{\partial z}+\frac{\partial Z}{\partial w}+\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right).$$

We may say that if $\mathrm{A}_2\,\mathrm{D}Q = 0$, Q is *leftward irrotational*, if $\mathrm{A}_2\,\mathrm{D}'\,Q' = 0$, then Q is *rightward irrotational*. The expressions A_2 we may speak of as *left curl* and *right curl*. If both vanish we may speak of Q as *irrotational*. The rotations here are of course planar and not axial. As an example of the use of these forms, let $Q = \varphi - A$ be a quaternion potential function, φ being a scalar potential, and A a vector potential. Then

$$\mathrm{A}_2\,\mathrm{D}Q = \nabla\varphi + \partial A/\partial w + \mathrm{V}\nabla A = -x + y,$$
$$\mathrm{A}_2\,\mathrm{D}'\,Q' = -\nabla\varphi - \partial A/\partial w + \mathrm{V}\nabla A = x + y.$$

Thence

$$2x = \mathrm{A}_2\,\mathrm{D}'\,Q' - \mathrm{A}_2\cdot\mathrm{D}Q, \qquad 2y = \mathrm{A}_2\,\mathrm{D}'\,Q' + \mathrm{A}_2\,\mathrm{D}Q.$$

From the identity $\mathrm{A}_3\,\mathrm{D}Q\mathrm{D} = 0$ we have, taking scalars, $\mathrm{S}\cdot\mathrm{VDV}Q\mathrm{VD} = 0$, or

$$\mathrm{S}\nabla y = 0.$$

Taking vectors we have

$$\mathrm{V}(\mathrm{A}_2\,\mathrm{D}Q\cdot\mathrm{D} + \mathrm{DA}_2\,\mathrm{D}'\,Q') = 0, \qquad \mathrm{VD}'(-x+y) + \mathrm{VD}(x+y) = 0,$$

that is,

$$\mathrm{V}\nabla x + \frac{\partial y}{\partial w} = 0.$$

Both equations are thus included in $\mathrm{A}_3\mathrm{D}Q\mathrm{D} = 0$, an identity.

Setting $2\mathrm{DA}_2\cdot Q'\mathrm{D}' = \mathrm{A}_2\,\mathrm{D}Q\cdot\mathrm{D} - \mathrm{DA}_2\,\mathrm{D}'\,Q' = 2q$, we have, by taking scalars,

$$\mathrm{S}\nabla x = -Sq.$$

† These invariants are coefficients of the characteristic equation; cf. JOLY, *Manual of Quaternions*.

Taking vectors we have

$$\mathrm{V}\nabla y + \frac{\partial x}{\partial w} = -\mathrm{V}q.$$

These two equations arise thus from a single equation.

Now if we set in these formulæ $w = \sqrt{-1}\cdot ct$, and in the first place set $x = -\sqrt{-1}\cdot E, y = H, Sq = \sqrt{-1}\cdot\rho$, we have the common equations of the electro-magnetic field.* If we set in the next place $x = -\sqrt{-1}\cdot E$, $y = B$, or $x = H$, $y = -\sqrt{-1}\cdot D$, respectively, we arrive at the vectors called by MINKOWSKI† electric force of rest, and magnetic force of rest,

$$R = \tfrac{1}{2}(\mathrm{A}_2\,\mathrm{D}Q\cdot r - r\mathrm{A}_2\,\mathrm{D}'\,Q'), \qquad S = \mathrm{A}_3\,\mathrm{D}Qr,$$

provided r is the velocity given by

$$\sqrt{-1}\cdot r = (1 - \sqrt{-1}\cdot q)/\sqrt{(1-q^2)}.$$

If we start with such equations as the familiar ones

$$\mathrm{V}\nabla H - \frac{\partial E}{\partial t} = \rho\sigma, \qquad \mathrm{S}\nabla E = -\rho, \qquad \mathrm{V}\nabla E + \frac{\partial H}{\partial t} = 0, \qquad \mathrm{S}\nabla H = 0,$$

we may reduce them to the present forms. We have first

$$\nabla H - \frac{\partial E}{\partial t} = \rho\sigma, \qquad -E\nabla + \frac{\partial H}{\partial t} = \rho.$$

Adding these, we have the single equivalent equation $\mathrm{D}H - E\mathrm{D} = q$. If $H = \mathrm{V}\nabla A$, and $E = \nabla\varphi + \partial A/\partial t$ this reduces with little trouble to

$$\mathrm{D}'\,\mathrm{I}\mathrm{D}'\,Q - \mathrm{I}\mathrm{D}\mathrm{D}'\cdot Q' = 2q.$$

Formulæ for the extensions of Stokes' theorem and Green's theorem will be found in the reference to Joly below.‡

When we operate on $\mathrm{T}q^{-2}$ or $(qq')^{-1}$ the operator $\mathrm{I}a\mathrm{D}$ will give the hyperspherical harmonics, and the results are susceptible of interpretations similar to those for three dimensions. The introduction of doublets follows analogous lines. These applications can only be mentioned.§

*Cf. LEWIS, Proceedings of the American Academy of Arts and Sciences, vol. 46 (1910), pp. 165–181; M. ABRAHAM, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1900), pp. 33–46. A. SOMMERFELD, Annalen der Physik und Chemie, ser. 4, vol. 32 (1910), pp. 749–776; vol. 33, pp. 649–689.

† MINKOWSKI, Göttinger Nachrichten, 1908, pp. 53, 111. The imaginary is not really necessary.

‡ Proceedings of the Royal Irish Academy, ser. 3, vol. 8 (1902), pp. 6–20.

§ Cf. JOLY, Appendix to HAMILTON'S *Elements of Quaternions*, 2d ed., Appendix 13. TAIT, *Scientific Papers*, vol. 1, 37–42, 134–135, 136–150, 176–193, 234–236, 280–281; vol. 2, p. 312; Quarterly Journal of Mathematics, vol. 6 (1864), pp. 279–301. MCAULAY, Messenger of Mathematics, vol. 14 (1884), pp. 26–37; Proceedings of the Royal Society of Edinburgh, vol. 18 (1890), pp. 98–123.

10. We pass now to the development of certain differential formulæ for three-dimensional curvilinear space, or what is the same thing, quaternions depending upon three parameters. This is equivalent to considering the three-dimensional space in a flat four-dimensional space, so that the problem becomes that of the differential geometry of curved three-spreads in four-dimensional space.* Let us suppose, then, that

$$q = f(u_1, u_2, u_3).$$

This equation will limit q to a certain three spread. We shall write

$$q_1 = \frac{\partial f}{\partial u_1}, \qquad q_2 = \frac{\partial f}{\partial u_2}, \qquad q_3 = \frac{\partial f}{\partial u_3}, \qquad dq = q_1\,du_1 + q_2\,du_2 + q_3\,du_3.$$

The quaternions q_1, q_2, q_3 are independent, as the parameters are essential. Let the quaternion $\mathrm{A}_3\,q_1\,q_2\,q_3 = n$. Then n is the normal at q to the three-flat tangent to the three-spread, since $\mathrm{I}nq_1 = 0$, $\mathrm{I}nq_2 = 0$, $\mathrm{I}nq_3 = 0$. The differential dq lies in the tangent three-flat.

11. We shall indicate the quaternions

$$\frac{-\mathrm{A}_3\,nq_2\,q_3}{\mathrm{I}nn}, \qquad \frac{-\mathrm{A}_3\,nq_3\,q_1}{\mathrm{I}nn}, \qquad \frac{-\mathrm{A}_3\,nq_1\,q_2}{\mathrm{I}nn}$$

respectively by $\bar{q}_1$, $\bar{q}_2$, and $\bar{q}_3$. These give at once

$$\mathrm{I}q_i\bar{q}_i = 1, \qquad \mathrm{I}q_i\bar{q}_j = 0 \quad (i \neq j), \qquad \mathrm{I}n\bar{q}_i = 0.$$

Any quaternion in the three-dimensional space may be expanded in the forms

$$r = q_1\,\mathrm{I}\bar{q}_1 r + q_2\,\mathrm{I}\bar{q}_2 r + q_3\,\mathrm{I}\bar{q}_3 r = q_1\,\mathrm{I}q_1 r + \bar{q}_2\,\mathrm{I}q_2 r + \bar{q}_3\,\mathrm{I}q_3 r.$$

We now define the operator

$$\Delta = \bar{q}_1\frac{\partial}{\partial u_1} + \bar{q}_2\frac{\partial}{\partial u_2} + \bar{q}_3\frac{\partial}{\partial u_3}.$$

We have by substitution of values the operator

$$\mathrm{I}dq\Delta = du_1\frac{\partial}{\partial u_1} + du_2\frac{\partial}{\partial u_2} + du_3\frac{\partial}{\partial u_3}.$$

That is to say, the operator $\mathrm{I}a\Delta$ gives the rate of variation of any operand due to a variation in the three-spread of q in the direction a, where a is a unit quaternion. It is evident again that if Q is a point function defined for the three-spread, a scalar let us say at first, then it will have certain level surfaces. The quaternion ΔQ gives us then the rate of maximum change, which will be the normal to a level surface. The operator Δ is thus the extension to curved

* Thus, if q is to terminate on a hypersphere, we might write

$$q = u_1\,i + u_2\,j + u_3\,k + \sqrt{(a^2 - u_1^2 - u_2^2 - u_3^2)}.$$

space of the operator ∇ for flat three-dimensional space. We proceed to show that it is invariant under transformations of the parameters. To do this let

$$u_1 = \varphi_1(v_1, v_2, v_3), \qquad u_2 = \varphi_2(v_1, v_2, v_3), \qquad u_3 = \varphi_3(v_1, v_2, v_3),$$

where the functions φ are differentiable and without singularities at the points considered. Then

$$\mathrm{A}_2 q_2 q_3 = \Sigma \mathrm{A}_2 q_{1v} q_{2v} \cdot \frac{\partial(v_1, v_2)}{\partial(u_2, u_3)}, \cdots \qquad \left(q_{1v} = \frac{\partial q}{\partial v_1}, \cdots\right).$$

Thence building up Δ_v we have $\Delta_v = \Delta_u$. Also we have

$$\mathrm{U}n_u = \mathrm{UA}_3 q_1 q_2 q_3 = \mathrm{UA}_3 (q_1)_v (q_2)_v (q_3)_v = \mathrm{U}n_v.$$

The normal is thus an invariant under transformation of the parameters. It follows naturally that such linear operators as $\mathrm{I}(\,)\Delta \cdot \mathrm{U}n$, $\mathrm{I}(\,)\Delta \cdot \Delta x$ and powers and combinations of these are also invariant. As an example, we see that the function $\mathrm{I}a\Delta \cdot Q$ gives the rate of variation of Q at the point in the direction a. The axes of this function are the directions of extremal variation and the roots are the rates of such change, at least if the function is self-transverse. If we lay off in each direction the value of the function (when Q is a scalar) the points so determined will have a hyper-quadric for their locus. The method of treatment differs little from the corresponding case for three dimensions.

Again consider the function $\mathrm{N} = \mathrm{I}(\,)\Delta \cdot \mathrm{U}n$. This is evidently the rate of change of the unit normal at the point. From the character of Δ it is self-transverse. Hence the axes are orthogonal and are the directions of the curvatures (reciprocals of the radii) at the point. The roots are real and are the curvatures themselves for real quaternions q_1, q_2, q_3. One root is zero, the corresponding axis being the normal. The usual scalar invariants of N are therefore the mean curvature, the second mean curvature, and the total or Kronecker-Gaussian curvature. The mean curvature (as in the similar three-dimensional case) is $\mathrm{I}\Delta\mathrm{U}n$. The second mean curvature and the total curvature are respectively

$$\mathrm{I} \cdot \mathrm{A}_3 n\Delta' \Delta'' \mathrm{A}_3 n^{-1} \mathrm{U}n' \mathrm{U}n'', \qquad \mathrm{IA}_3 \Delta' \Delta'' \Delta''' \mathrm{A}_3 \mathrm{U}n' \mathrm{U}n'' \mathrm{U}n''',$$

where the accents are dropped after the expansion.

12. Since Δu, Δv, Δw, $\cdots$ are invariant, also $\mathrm{T}\Delta u$, $\mathrm{T}\Delta v$, $\cdots$ are invariant. Indeed it is evident that the differential parameters of the first order for the functions u, v, $\cdots$ are, in the usual notation for differential parameters,

$$\Delta_1 u = \mathrm{I}\Delta u \Delta u, \qquad \Delta_1(u, v) = \mathrm{I}\Delta u \Delta v.$$

In addition to these we have the following extensions of the Darboux $\Theta(u, v)$

for two dimensions to three dimensions:

$$\mathrm{A}_2\,\Delta u\Delta v, \qquad \mathrm{A}_1\,\Delta u\Delta v\Delta w, \qquad \mathrm{A}_3\,\Delta u\Delta v\Delta w,$$

giving the respective parameters

$$\mathrm{IA}_2\,\Delta u\Delta v\mathrm{A}_2\,\Delta u\Delta v, \qquad \mathrm{I}\cdot\mathrm{A}_1\,\Delta u\Delta v\Delta w\mathrm{A}_1\,\Delta u\Delta v\Delta w, \qquad \mathrm{IA}_3\,\Delta u\Delta v\Delta w\mathrm{A}_3\,\Delta u\Delta v\Delta w,$$

$$\mathrm{I}\Delta u\Delta v\Delta w\Delta x, \qquad \mathrm{IA}_2\,\Delta u\Delta v\Delta w\Delta x\mathrm{A}_2\,\Delta u\Delta v\Delta w\Delta x.$$

The form $\mathrm{A}_4\,\Delta u\Delta v\Delta w\Delta x$ vanishes identically.
The formulæ of part I give various syzygies connecting these, for example

$$\mathrm{I}\cdot\Delta u\Delta v\Delta w\Delta x = \Delta_1(u, v)\,\Delta_1(w, x) - \Delta_1(u, w)\,\Delta_1(v, x) + \Delta_1(u, x)\,\Delta_1(v, w).$$

Since Δu is the normal in the three-spread for the levels of the scalar function u, the parameter of first order of u, $\Delta_1 u$, is the square of the length of this normal. The differential parameter of first order of u and v is the projection of the normal to the level of either, on the other, multiplied by the length of the latter. If it vanishes the levels are orthogonal. The interpretation of the other forms follows the exactly analogous forms for ∇ in a flat three-dimensional space.

13. Any parameter of the first order is itself subject to operation and we arrive thus at compound forms such as $\mathrm{I}\Delta u\Delta(\Delta_1 u)$. We are led thus to consider operators such as $\Phi = \mathrm{I}(\,)\Delta\cdot\Delta u$. The first scalar invariant of this linear quaternion operator is $\mathrm{I}\Delta\Delta u$, which is the well-known differential parameter $\Delta_2 u$. It vanishes if u is a solution of the generalized Laplacian equation. Indeed if the space becomes flat and the position is determined by ρ instead of q, the second differential parameter becomes $-\nabla^2$. If the lines u_1, u_2, u_3 are isothermal, we have a simplification which is not difficult to follow. The second invariant of Φ is

$$\mathrm{IA}_3\,n\Delta'\Delta''\,\mathrm{A}_3\,n^{-1}(\Delta u)'(\Delta u)''.$$

This can be expanded in the usual manner for the coefficients of the characteristic equation of a matrix or linear quaternion operator, in the form

$$\tfrac{1}{2}(\mathrm{I}\Delta\Delta u)^2 - \mathrm{I}\Delta'\Delta''u\mathrm{I}\Delta'\Delta''u.$$

It is the well-known parameter $\Delta_{22} u$. Finally the third invariant is

$$-\tfrac{1}{6}\,\mathrm{IA}_3\,\Delta'\Delta''\Delta'''\,\mathrm{A}_3\,(\Delta u)'(\Delta u)''(\Delta u)'''.$$

This might be called the parameter $\Delta_{222} u$. The fourth invariant vanishes identically.

The function Φ is the rate of change in any direction of the vector rate of change of the function which is itself the normal to a level of the function.

14. If we consider only the direction of the normal of the level, we have the function $\Xi = \mathrm{I}(\,)\Delta \cdot \mathrm{U}\Delta u$, which gives the rate of deviation of the normal direction. There are now only two invariants that do not vanish identically, and these determine what may be called geodetic curvatures. The first invariant is $\mathrm{I}\Delta\mathrm{U}\Delta u$, the geodetic mean curvature. The second is the expression

$$\mathrm{IA}_3\, n\Delta'\,\Delta''\,\mathrm{A}_3\, n^{-1}\,\mathrm{U}\Delta u'\,\mathrm{U}\Delta u''.$$

These are the extensions of the usual geodetic curvature for two dimensions.

15. We notice next the fundamental forms or covariants of different orders. The first is

$$\mathrm{I}dqdq = \Sigma\mathrm{I}q_1\, q_1\, du_1\, du_1 + 2\Sigma\mathrm{I}q_1\, q_2\, du_1\, du_2.$$

The coefficients $\mathrm{I}q_i\, q_j$ correspond to the E, F, G of the geometry of surfaces. They are expressible as parameters, for example,

$$\mathrm{I}q_1\, q_2 = \mathrm{IA}_3\, n\Delta u_2\, \Delta u_3\, \mathrm{A}_3\, n\Delta u_3\, \Delta u_1.$$

Other covariant forms are

$$\mathrm{I}dqdr,\quad \mathrm{A}_2\, dqdr,\quad \mathrm{A}_1\, dqdrds,\quad \mathrm{A}_3\, dqdrds,\quad \mathrm{I}dqdrdsdt,\quad \mathrm{A}_2\, dqdrdsdt,$$

$$\mathrm{A}_2\, ndq,\quad \mathrm{A}_1\, ndqdr,\quad \mathrm{A}_3\, ndqdr,\quad \mathrm{A}_2\, ndqdrds,\quad \mathrm{A}_4\, ndqdrds \equiv 0.$$

Denoting differentiation as to u_i by the subscript i we have, from $\mathrm{IU}n\mathrm{U}n = 1$,

$$\mathrm{IU}n\,(\mathrm{U}n)_i = 0,$$

$$\mathrm{IU}n\,(\mathrm{U}n)_i = 0, \qquad \mathrm{IU}n\,(\mathrm{U}n)_{ij} = -\,\mathrm{I}\,(\mathrm{U}n)_i\,(\mathrm{U}n)_j \quad (i, j = 1, 2, 3).$$

Since $\mathrm{I}nq_i = 0$, therefore

$$\mathrm{I}nq_{ij} = -\,\mathrm{I}n_iq_j = -\,\mathrm{I}n_jq_i.$$

In this we might also have written $\mathrm{U}n$ instead of n. We arrive thus at the extensions of the D, D′, D″, of two dimensions. The Christoffel symbols are

$$\begin{bmatrix} kl \\ i \end{bmatrix} = \mathrm{I}q_iq_{kl}, \qquad \begin{Bmatrix} kl \\ i \end{Bmatrix} = \mathrm{I}q_{kl}\Delta u_i = \mathrm{I}q_{kl}\bar{q}_i.$$

$$(ikrs) = \mathrm{IA}_3\, n^{-1}\,\mathrm{U}n_i\mathrm{U}n_k\mathrm{A}_3nq_rq_s.$$

Differentiating these and remembering that also Δ is subject to differentiation, we arrive at the symbols of higher order. This is the covariantive differentiation of Maschke.*

The second fundamental form is

$$\mathrm{I}dqd\mathrm{U}n = \mathrm{I}dq\mathrm{N}dq = \Sigma\mathrm{I}q_1\,(\mathrm{U}n)_1\, du_1\, du_1 + 2\Sigma\mathrm{I}q_i\,(\mathrm{U}n)_j\, du_i\, du_j.$$

* Maschke, loc. cit. Also these Transactions, vol. 4 (1903), pp. 457.

This form vanishes for the asymptotic directions, or the principal tangent lines of the three-spread. It is worth noting that the differential equation of the lines of curvature is $A_2\, dqNdq = 0$. This corresponds to the similar form for surfaces $Vd\rho\varphi d\rho = 0$.

16. The symbolic invariants of Maschke are easily expressed in these forms by noticing the following equivalences* for his symbols

$$f_i = I\zeta q_i, \qquad f_i f_k = I\zeta q_i I\zeta q_k = Iq_i q_k, \qquad f_{kl} = I\zeta q_{kl}.$$

$$(f) = A_4\, Un\zeta_1\, \zeta_2\, \zeta_3,$$

whence

$$(f)^2 = 3!\, IUnUn = 3!, \qquad (uf) = A_4\ n\Delta u\zeta_1\, \zeta_2.$$

$$(fu) = A_4\, Un\Delta u'\, \Delta u''\, \zeta_1,$$

whence

$$(uf)^2 = 2I\Delta u\Delta u = 2\Delta_1 u.$$

$$((uf), f) = A_4\, Un\Delta\, (A_4\, Un\Delta u\zeta_1\, \zeta_2)\, \zeta_1\, \zeta_2 = 2IUnUnI\Delta\Delta u = 2\Delta_2 u.$$

Not to elaborate too far, we need only to remember that each ( ) is an A_4, that u, v, w, x, etc., are functions of q such as we have been using, that for every f or like symbol we substitute $I\zeta q$, for subscripts we differentiate as to the corresponding u. All the Maschke symbols for four dimensions become at once interpretable in this system. The vectors of INGOLD † are the quaternions appearing in these forms, his formulæ being analogues of Maschke's, not equivalents. The ordinary formulæ hold of course for these quaternions, but the Maschke forms are all scalars and commutative, and are the scalar forms with the ζ-pairs introduced.

* The bilinear function $Q(\xi, \zeta)$ is an abbreviation for $Q(q_1, \bar{q}_1) + Q(q_2, \bar{q}_2) + Q(q_3, \bar{q}_3)$. We may always write for a bilinear function $Q(\zeta, \zeta)$ the form $Q(\Delta, q)$ or equally $Q(q, \Delta)$. See MCAULAY, *Utility of Quaternions in Physics;* SHAW, *Synopsis of Linear Associative Algebra.*

† These Transactions, vol. 11 (1910), pp. 449–474.

THEORY OF FINITE ALGEBRAS*

BY

H. S. VANDIVER

1. The object of this paper is to develop an abstract theory of finite algebras which is applicable to various familiar and important concrete algebras, such as the m classes of residues of integers modulo m, the classes of residues of polynomials in x with respect to the moduli m and $P(x)$, the classes of residues of integral algebraic numbers of a given algebraic field with respect to any ideal as modulus, and the classes of residues with respect to certain modular systems. We consider an algebra composed of a finite set of elements which may be combined by addition, subtraction and multiplication, subject to the commutative, associative and distributive laws, and such that the sum, difference or product of any two elements is uniquely determined as an element of the set, while, moreover, there occurs an element playing the rôle of unity under multiplication. It is not assumed that division is always possible; a product may vanish when neither factor vanishes.

While the field $R(\alpha_1, \cdots, \alpha_n)$ defined by the algebraic numbers $\alpha_1, \cdots, \alpha_n$ is identical with a field $R(u)$ defined by a single algebraic number u, a similar theorem does not hold for finite algebras (§ 6).

As regards their units, primes, etc., the theory of finite algebras presents analogies with the theory of integral algebraic numbers.

I am indebted to Professor L. E. DICKSON for valuable suggestions.

2. **Definition of a Finite Algebra.** Let the elements $u_0, u_1, \cdots, u_{s-1}$ form a commutative group under addition. The unique element u_0 such that $u_i + u_0 = u_i$ $(i = 0, \cdots, s-1)$ is called the zero element. It is assumed that the elements may be combined by multiplication, subject to the commutative, associative and distributive laws, and that the product of any two elements is an element of the set. We shall discard the assumption, made in the theory of finite fields, that division by every element other than u_0 is possible and unique. However, we shall assume that there exists at least one element u_k, called a *unit element*, such that $u_k x = u_0$ has the unique solution $x = u_0$. Elements which are not units are called *non-units* and denoted by $N_1, N_2, \cdots$.

Theorem I. *Division by a unit U_k is always possible and unique.*

Since the products $u_i U_k$ $(i = 0, 1, \cdots, s-1)$ are distinct, they form a permutation of the u_i. Thus $xU_k = u_l$ has one and but one solution x.

* Presented to the Society (Chicago), December 29, 1911.

The solution x of $xU_k = U_k$ plays the rôle of *unity*. For, if y is the solution of $yU_k = u_l$, then

$$xu_l = (xU_k)\, y = U_k\, y = u_l.$$

Theorem II. *The product of two or more units is a unit.*

For, if $U_1 U_2 \cdots U_k = N_1$, we can determine a non-unit $N_2 \neq u_0$ such that $N_1 N_2 = u_0$. Then

$$U_1 (N_2 U_2 \cdots U_k) = u_0,$$

so that

$$N_2 U_2 \cdots U_k = u_0.$$

A repetition of the process gives $N_2 = u_0$.

Theorem III. *A product is a non-unit if any factor is.*

By Theorem II such a product may be given the form

$$P = UN_1 N_2 \cdots N_e \qquad (U \text{ a unit}).$$

Set $N_1 N_k = u_0$, $N_k \neq u_0$. Then $PN_k = u_0$, so that P is a non-unit.

Theorem IV. *The totality of units $U_1, \cdots, U_k$ forms a commutative group under multiplication; the kth power of any unit is unity.*

The products $U_i U_1, \cdots, U_i U_k$ are distinct and each is a unit (Theorem II). Hence they form a permutation of $U_1, \cdots, U_k$, so that $U_i x = U_j$ has as its unique solution a unit.

By Thorems I and III, every element is divisible by each unit, and no unit is divisible by a non-unit. But a given non-unit may or may not be divisible by another given non-unit. Accordingly, we shall make the

Definitions. If N is a non-unit, any element UN is called an *associate* of N, where U is a unit. A non-unit is called a *prime* if it has no non-unit divisor other than itself and its associates. Two or more non-units are called *prime to each other* if no two of them have a common non-unit divisor.

3. Consider the algebra $A(m)$ formed of the m classes of residues C_r of integers modulo m, where $C_r = C_\rho$ if and only if $r \equiv \rho \pmod{m}$, and

$$C_r \pm C_\rho = C_{r \pm \rho}, \qquad C_r\, C_\rho = C_{r\rho}.$$

Then C_0 is the zero element and C_1 the unity. The units are C_a, where a ranges over the integers less than m and prime to m. The non-units are C_r, where r is an integer $< m$ having a factor in common with m. If e and f are relatively prime integers, we have $1 = eu + fv$, where u and v are integers. Hence

$$C_1 = C_e\, C_u + C_f\, C_v,$$

so that any common divisor of C_e and C_f divides C_1 and hence (Theorem III) is a unit. Thus C_e and C_f are prime to each other.

If an element of $A(m)$ is a prime it is the associate of an element C_p,

where p is a prime factor of m. If C_p is composite we must have

$$C_p = UC_{p^\epsilon} \qquad (\epsilon > 1),$$

where U is a unit. Let p^α be the highest power of p which divides m. If $\epsilon < \alpha$ we multiply the members of the preceding equation by C_{m/p^ϵ} and obtain $C_{m/p^{\epsilon-1}} = C_0$, whence $\epsilon = 1$. If $\epsilon \geqq \alpha$, we multiply by C_{m/p^α} and obtain $C_{m/p^{\alpha-1}} = C_0$, whence $\alpha = 1$. Then

$$s = p\left(p^{\epsilon-1} - \frac{m}{p}\right) \equiv p^\epsilon \pmod{m}$$

and C_s is an associate of C_p. Hence C_p is not composite.

Theorem V. *If $p, q, \cdots, r$ are the distinct prime factors of m, the algebra $A(m)$ composed of the m classes of residues of integers modulo m contains the prime elements $C_p, C_q, \cdots, C_r$; these and their associates give all the prime elements.*

Any finite algebra $A(s)$ contains * a sub-algebra $A(m)$ composed of the integral elements C_r. We shall call m the base of $A(s)$. If e and f are relatively prime integers, C_e is prime to C_f within $A(s)$.

We note, in addition to $A(m)$, the following examples of finite algebras. Miss SANDERSON † has considered the classes of residues

$$a(y) = a_0 + a_1 y + \cdots + a_{r-1} y^{r-1} \quad (a_i = 0, 1, \cdots, m-1)$$

modulis $m, P(y)$, where m is any integer and $P(y)$ is a polynomial in y of degree r with integral coefficients, irreducible with respect to each prime factor of m as modulus. She proved that $a(y)$ is a unit element if and only if the greatest common divisor of its coefficients a_i is prime to m, and that number of units is

$$m^r\left(1 - \frac{1}{p_1^r}\right)\left(1 - \frac{1}{p_2^r}\right)\cdots\left(1 - \frac{1}{p_k^r}\right),$$

where $p_1, \cdots, p_k$ are the distinct prime factors of m.

DEDEKIND ‡ has considered the classes of residues of integral numbers of an algebraic field with respect to an ideal α as modulus. These classes form a finite algebra the number of unit elements of which is

$$N(\alpha)\left\{1 - \frac{1}{N(\rho_1)}\right\}\cdots\left\{1 - \frac{1}{N(\rho_r)}\right\},$$

where $\rho_1, \cdots, \rho_r$ are the distinct prime ideal factors of α, and $N(k)$ is the norm of k. In particular, if α is a prime ideal of norm p^f, the classes form a finite field of order $s = p^f$.

* L. E. DICKSON, *Linear Groups* (Teubner), 1901, p. 9.

† Annals of Mathematics, (2), vol. 13 (1911), p. 36.

‡ DIRICHLET-DEDEKIND, *Zahlentheorie*, 4th ed., pp. 564–580.

4. **Congruence.** Let $M_1, \cdots, M_t$ be elements of a finite algebra $A(s)$. Two elements A and B of the algebra are called congruent (and residues of each other) with respect to a modular system $(M_1, \cdots, M_t)$ if

$$A - B = k_1 M_1 + \cdots + k_t M_t,$$

where $k_1, \cdots, k_t$ are elements of the algebra. We then write

$$A \equiv B \qquad (\text{modd } M_1, \cdots, M_t).$$

All the elements congruent to a given one constitute a class of residues. The classes of residues which give without duplication all the elements form a complete system of classes of residues.

Consider the case of a single modulus M. If an element A is such that $Ax \equiv 1 \pmod{M}$ holds for an element x of the algebra, A is called a *unit residue* modulo M. If $A_1, \cdots, A_k$ are the incongruent unit residues modulo M, the products $A_i A_1, \cdots, A_i A_k$ are incongruent and hence have in some order the residues $A_1, \cdots, A_k$ modulo M. Thus the unit residues form a commutative group. Any element of the algebra is divisible modulo M by any unit residue.

Order of a finite algebra.

5. The integral marks of the algebra will henceforth be denoted by $0, 1, \cdots, m-1$. Let

$$m = p^\alpha q^\beta \cdots r^\gamma, \tag{1}$$

where $p, \cdots, r$ are distinct primes. Then m/p^α is used to denote $q^\beta \cdots r^\gamma$ and not to denote a division to be performed within the algebra. Let S be any element of the algebra; set

$$S \equiv P \pmod{p^\alpha}, \qquad S \equiv Q \pmod{q^\beta}, \cdots, \qquad S \equiv R \pmod{r^\gamma}.$$

Since the power $\varphi(p^\alpha)$ of m/p^α is $\equiv 1$ modulo p^α and to zero modulo $q^\beta, \cdots$, or r^γ, we have in the algebra

$$\left(\frac{m}{p^\alpha}\right)^{\phi(p^\alpha)} S = \left(\frac{m}{p^\alpha}\right)^{\phi(p^\alpha)} P,$$

since the two are congruent modulo m. Adding this equation to those derived by replacing p by $q, \cdots, r$ in turn, we get

$$S = \left(\frac{m}{p^\alpha}\right)^{\phi(p^\alpha)} P + \left(\frac{m}{q^\beta}\right)^{\phi(q^\beta)} Q + \cdots + \left(\frac{m}{r^\gamma}\right)^{\phi(r^\gamma)} R, \tag{2}$$

the coefficient of S being initially the sum of those of $P, \cdots, R$ and hence $\equiv 1$ with respect to each modulus $p^\alpha, \cdots, r^\gamma$ and hence with respect to m. When $P, \cdots, R$ range over the distinct residues of the elements modulo

$p^\alpha, \cdots, r^\gamma$, respectively, the sum (2) ranges without repetition over the elements of the algebra. For, if (2) equals the same function of $P', \cdots, R'$, then

$$P \equiv P' \pmod{p^\alpha}, \quad \cdots, \quad R \equiv R' \pmod{r^\gamma}.$$

Theorem VI. *If $N(p^\alpha)$ is the number of residues modulo p^α of the elements of the algebra, the order of the algebra is*

$$N(p^\alpha) \cdot N(q^\beta) \cdots N(r^\gamma). \tag{3}$$

To evaluate $N(p^\alpha)$ we first exhibit a set of multiples of $p^{\alpha-1}$ incongruent modulo p^α such that every element which is a multiple of $p^{\alpha-1}$ is congruent to one of the set modulo p^α. The set contains

$$p^{\alpha-1}(d_1 v_1) \qquad (d_1 = 0, 1, \cdots, p-1; v_1 = 1).$$

If $p^{\alpha-1} v_2$ is an additional element of the set, then

$$p^{\alpha-1}(d_1 v_1 + d_2 v_2) \qquad (d_1, d_2 = 0, 1, \cdots, p-1)$$

are easily seen to be incongruent modulo p^α. Proceeding in this way we see that the set is exhausted by the elements, incongruent modulo p^α,

$$p^{\alpha-1} \sum_{k=1}^{e_{\alpha-1}} d_k v_k \qquad (d_k = 0, 1, \cdots, p-1).$$

It follows at once that

$$p^{\alpha-2} \sum_{k=1}^{e_{\alpha-2}} d_k v_k \qquad (d_k = 0, 1, \cdots, p-1)$$

are incongruent modulo $p^{\alpha-1}$. If $p^{\alpha-2} v_f$, where $f = e_{\alpha-1} + 1$, is not congruent to one of the preceding modulo $p^{\alpha-1}$, then

$$p^{\alpha-2} \sum_{k=1}^{f} d_k v_k \qquad (d_k = 0, 1, \cdots, p-1)$$

are incongruent modulo $p^{\alpha-1}$. In this way we see that all multiples of $p^{\alpha-2}$ incongruent modulo $p^{\alpha-1}$ are given without repetition by

$$p^{\alpha-2} \sum_{k=1}^{e_{\alpha-2}} d_k v_k \qquad (d_k = 0, 1, \cdots, p-1; e_{\alpha-2} \geqq e_{\alpha-1}). \tag{4}$$

To obtain a complete set of incongruent residues modulo p^α of the multiples of $p^{\alpha-2}$ we have merely to add to the elements (4) a complete set of multiples of $p^{\alpha-1}$ incongruent modulo p^α:

$$p^{\alpha-2} \sum_{k=1}^{e_{\alpha-2}} d_k v_k + p^{\alpha-1} \sum_{k=1}^{e_{\alpha-1}} d_k v_k \quad (d_k = 0, 1, \cdots, p-1).$$

The multiples of $p^{\alpha-\alpha}$ include all elements. Hence after α steps we obtain a

complete set of incongruent residues modulo p^a:

$$\sum_{c=0}^{a-1}\left(p^c \sum_{k=1}^{e_c} d_{ck}\, v_k\right) \quad (d_{ck} = 0, 1, \cdots, p-1; e_{c-1} \geqq e_c). \tag{5}$$

Hence $N(p^a) = p^\pi$, where $\pi = e_0 + e_1 + \cdots + e_{a-1}$. Thus $\pi \leqq \alpha$.

Theorem VII. *The order of a finite algebra is a multiple of its base m and contains no prime factor not occurring in m.*

6. Consider an element u of a finite algebra and let n be the least positive integer for which u^n is a linear function of $u^{n-1}, \cdots, u$ with integral coefficients. In case u can be chosen so that the elements

$$a_0 + a_1 u + \cdots + a_{n-1} u^{n-1} \quad (a_j = 0, 1, \cdots, m-1) \tag{6}$$

exhaust all elements of the algebra, the latter is called a monomorphic algebra. We give an example of an algebra not of this type.

Consider the classes of residues with respect to the modular system

$$(x^4, y^4, p)$$

x and y being arbitrary variables and p a prime. There are p^{16} distinct classes of residues represented by

$$\sum_{a,b}^{0,1,2,3} c_{a,b}\, x^a y^b \qquad (c_{ab} = 0, 1, \cdots, p-1).$$

Suppose that this algebra is monomorphic. Then if u is a properly chosen one of the preceding residues, and if n is the least positive integer for which

$$u^n + c_1 u^{n-1} + \cdots + c_n \equiv 0 \pmod{x^4, y^4, p},$$

$c_1, \cdots, c_n$ being integers, we conclude from the fact that $m = p$ is a prime that no two of the elements (6) are congruent with respect to (x^4, y^4, p). By hypothesis, every residue may be given the form (6). Hence $n = 16$.

The p^4 incongruent residues with respect to (x^2, y^2, p) are congruent to certain elements (6) with respect to (x^4, y^4, p) and hence with respect to (x^2, y^2, p). But if k is the least integer for which

$$\psi(u) = u^k + d_1 u^{k-1} + \cdots + d_k \equiv 0 \pmod{x^2, y^2, p},$$

with $d_1, \cdots, d_k$ integers, there are exactly p^k polynomials in u, with integral coefficients, incongruent with respect to (x^2, y^2, p). Hence $k = 4$. From $\psi(u) = \alpha x^2 + \beta y^2 + \gamma p$, we get

$$u^{12} + \cdots = \{\psi(u)\}^3 \equiv 0 \pmod{x^4, y^4, p}.$$

Since this contradicts $n = 16$, the algebra is not monomorphic.

We employ the notation (1) for m and make the definition:

For $k < \alpha$, the least integer e_k for which

$$p^k (u^{e_k} + c_1 u^{e_k-1} + \cdots) \equiv 0 \pmod{p^\alpha} \qquad (c\text{'s integers}) \tag{7}$$

is called the degree of u modulo p^α, coefficients p^k. In particular, e_0 is called the degree of u modulo p^α. Multiplying

$$p^{k-1} (u^{e_{k-1}} + c_1' u^{e_{k-1}-1} + \cdots) \equiv 0 \pmod{p^\alpha}$$

by p and comparing the result with (7), we get $e_k \leqq e_{k-1}$.

Theorem VIII. *If e_k is the degree of u modulo p^α, coefficient p^k, it is also the degree of u modulo p^ϵ, coefficient p^k, where $\alpha > \epsilon > k$.*

Let s the least positive integer for which

$$p^k (u^s + d_1 u^{s-1} + \cdots) \equiv 0 \pmod{p^\epsilon} \qquad (d\text{'s integers}).$$

Since u^{e_0} is congruent modulo p^α to a sum of lower powers of u,

$$p^k (u^s + d_1 u^{s-1} + \cdots) \equiv p^\epsilon \sum_{t=1}^{e_0} f_t u^{e_0-t} \pmod{p^\alpha},$$

where the f_t are integers. If $s < e_k$, multiply each member by the power $e_k - 1 - s$ of u and reduce exponents of u by means of (7). Thus

$$p^k \{ (1 - p^{\epsilon-k} c) u^{e_k-1} + c' u^{e_k-2} + \cdots \} \equiv 0 \pmod{p^\alpha}.$$

Multiply this by g, where

$$g (1 - p^{\epsilon-k} c) \equiv 1 \pmod{p^\alpha}.$$

We obtain a congruence contradicting the definition of e_k in (7).

Order of a monomorphic algebra.

7. Let every element be a polynomial (6) in u. Since u is of degree e_0 modulo p^α, any residue modulo p^α may be written

$$\sum_{\epsilon=0}^{\alpha-1} p^\epsilon \sum_{k=1}^{e_0} d_{\epsilon k} u^{e_0-k} \qquad (0 \leqq d \leqq p-1). \tag{8}$$

The terms with $\epsilon = 1$ may be reduced by (7) for $k = 1$, to the form

$$p \sum_{k=1}^{e_1} f_{1k} u^{e_1-k} + p^2 F_2(u) + \cdots + p^{\alpha-1} F_{\alpha-1}(u),$$

where the f's and the coefficients of the F's belong to the set $0, 1, \cdots, p-1$ and each F is of degree $< e_1$. In the new form of (8) occurs

$$p^2 \left\{ \sum_{k=1}^{e_0} d_{2k} u^{e_0-k} + F_2(u) \right\},$$

which may be reduced by (7) for $k = 2$ to the form

$$p^2 \sum_{k=1}^{e_2} f_{2k} u^{e_2-k} + p^3 G_2(u) + \cdots + p^{a-1} G_{a-1}(u).$$

In this way any residue modulo p^a can be given the form

$$\sum_{c=0}^{a-1} p^c \sum_{k=1}^{e_c} f_{ck} u^{e_c-k} \qquad (0 \leqq f \leqq p-1). \tag{9}$$

Theorem IX. *These elements* (9) *are all incongruent modulo* p^a.

For, if

$$\sum_{c=0}^{a-1} p^c \sum_{k=1}^{e_c} (f_{ck} - g_{ck}) u^{e_c-k} \equiv 0 \pmod{p^a}, \tag{10}$$

where each f and g is one of the integers $0, 1, \cdots, p-1$, then

$$\sum_{k=1}^{e_0} (f_{0k} - g_{0k}) u^{e_0-k} \equiv 0 \pmod{p}.$$

By Theorem VIII if $\alpha > 1$ or by (7) if $\alpha = 1$, we get

$$f_{0k} - g_{0k} \equiv 0 \pmod{p} \qquad (k = 1, \cdots, e_0).$$

Hence $f_{0k} = g_{0k}$. The theorem is proved if $\alpha = 1$. If $\alpha > 1$, (10) gives

$$p \sum_{k=1}^{e_1} (f_{1k} - g_{1k}) u^{e_1-k} \equiv 0 \pmod{p^2}.$$

By Theorem VIII if $\alpha > 2$ or by (7) if $\alpha = 2$,

$$f_{1k} - g_{1k} \equiv 0 \pmod{p}, \qquad f_{1k} = g_{1k}.$$

A repetition of this process proves the theorem for any α.

Write $e_k^{(p)}$ for e_k. Similarly, let $e_k^{(q)}$ be the degree of u modulo q^β, coefficient q^k, etc. Then by (3) and (9), we obtain

Theorem X. *The order of the algebra is* $p^\pi \cdots r^\rho$, *where*

$$\pi = \sum_{k=0}^{a-1} e_k^{(p)}, \cdots, \qquad \rho = \sum_{k=0}^{\gamma-1} e_k^{(r)}.$$

Units of a monomorphic algebra.

8. The degree e_0 of u modulo p^a is also the degree of u modulo p (Theorem VIII). Let therefore

$$F(u) = u^{e_0} + c_1 u^{e_0-1} + \cdots + c_{e_0} \equiv 0 \pmod{p^a},$$
$$F(u) \equiv g_1^{k_1} g_2^{k_2} \cdots g_s^{k_s} \pmod{p},$$

where $g_1, \cdots, g_s$ are incongruent irreducible polynomials in u modulo p,

and g_i is of degree n_i. Thus $\Sigma k_i n_i = e_0$. The p^{e_0} elements

$$\sum_{i=1}^{s} \frac{F(u)}{g_i^{k_i}} \sum_{\epsilon=1}^{n_i k_i} c_{i\epsilon} u^{n_i k_i - \epsilon} \qquad (c = 0, 1, \cdots, p-1) \tag{11}$$

are incongruent modulo p. For, if not, we have, for example,

$$\frac{F(u)}{g_1^{k_1}} \sum_{\epsilon=1}^{n_1 k_1} (c_{1\epsilon} - c'_{1\epsilon}) u^{n_1 k_1 - \epsilon} \equiv 0 \pmod{g_1^{k_1}, p}, \tag{12}$$

in which $c_{1\epsilon} - c'_{1\epsilon}$ is not divisible by p for every ϵ. There exists * a polynomial $G(u)$ with integral coefficients such that in the algebra

$$G(u) \cdot \frac{F(u)}{g_1^{k_1}} \equiv 1 \pmod{g_1^{k_1}, p}.$$

Multiply (12) by $G(u)$. Hence

$$\sum_{\epsilon=1}^{n_1 k_1} (c_{1\epsilon} - c'_{1\epsilon}) u^{n_1 k_1 - \epsilon} \equiv L g_1^{k_1} \pmod{p}.$$

Multiply this by $F(u)/g_1^{k_1}$. Thus

$$\psi(u) \equiv LF(u) \equiv 0 \pmod{p},$$

where $\psi(u)$ is of degree $< e_0$, contrary to the fact that u is of degree e_0 modulo p. Hence *the p^{e_0} elements* (11) *give a complete set of residues modulo p.*

The residue of any element modulis $g_1^{k_1}, p$ may be written

$$\sum_{i=0}^{k_1-1} g_1^i \sum_{\gamma=1}^{n_1} d_{i\gamma} u^{n_1 - \gamma} \qquad (0 \leqq d \leqq p-1). \tag{13}$$

No two of these are congruent modulis $g_1^{k_1}, p$. For, if (13) be congruent to the element with the coefficients d', we have

$$\sum_{\gamma=1}^{n_1} (d_{0\gamma} - d'_{0\gamma}) u^{n_1 - \gamma} \equiv 0 \pmod{g_1, p}.$$

Unless each coefficient is divisible by p (and hence zero), we obtain by multiplication by $F(u)/g_1$ a congruence modulo p of degree $< e_0$. Thus $d_{0\gamma} = d'_{0\gamma}$ for each γ. Then a similar argument with

$$g_1 \sum_{\gamma=1}^{n_1} (d_{1\gamma} - d'_{1\gamma}) u^{n_1 - \gamma} \equiv 0 \pmod{g_1^2, p}$$

shows that $d_{1\gamma} = d'_{1\gamma}$ for each γ; etc. *The elements* (13) *form a complete set of residues modulis $g_1^{k_1}, p$.*

Let R be an element (13) in which not every $d_{0\gamma} = 0$. Then R and $g_1^{k_1}$ have no common divisor modulo p involving u. Hence† there exist two

* DICKSON, *Linear Groups*, p. 8.

† DICKSON, *Linear Groups*, p. 8.

polynomials K and L in u with integral coefficients such that

$$KR - Lg_1^{k_1} \equiv 1 \pmod{p}.$$

By the definition in § 4, R is a unit residue modulis $g_1^{k_1}, p$.

If an element $g_1 R$ of the form (13) with each $d_{0\gamma} = 0$ were a unit residue, $g_1 RK \equiv 1 \pmod{g_1^{k_1}, p}$, we would have $F(u)/g_1 \equiv 0 \pmod{p}$.

An element (13) *is a unit residue modulis* $g_1^{k_1}, p$ *if and only if not every* $d_{0\gamma}$ *is zero. The number of these unit residues is*

$$p^{n_1 k_1}\left(1 - \frac{1}{p^{n_1}}\right).$$

It now follows from (11) that

$$U = \sum_{i=1}^{s} \frac{F(u)}{g_i^{k_i}} R_i \tag{14}$$

ranges over the incongruent residues modulo p prime to $g_1, \cdots, g_s$ when R_i ranges over the unit residues modulis $g_i^{k_i}, p$. We can determine two functions H and M of u such that

$$HU - MF(u) \equiv 1 \pmod{p},$$

whence $HU \equiv 1 \pmod{p}$. Thus U is a unit residue modulo p.

Theorem XI. *The number of unit residues modulo* p *is*

$$p^{e_0} \prod_{i=1}^{s}\left(1 - \frac{1}{p^{n_i}}\right).$$

To apply this result to the enumeration of the units, we require the

Lemma. *If two polynomials* $f_0(z)$ *and* $f_1(z)$ *with integral coefficients have no common factor involving* z *with respect to any of the prime factors of* m *as modulus, there exist two polynomials* $\psi_0(z)$ *and* $\psi_1(z)$ *with integral coefficients such that*

$$f_0 \psi_0 - f_1 \psi_1 \equiv 1 \pmod{m}. \tag{15}$$

We first prove the lemma when m is a power p^a of a prime. There exist polynomials $x(z)$, $y(z)$ and $w(z)$ such that

$$xf_0 - yf_1 = 1 - pw.$$

Multiply this by $1 + pw + p^2w^2 + \cdots + (pw)^{a-1}$. We obtain (15).

Next, let

$$m = \Pi p_i^{a_i}, \qquad f_0 x_i - f_1 y_i \equiv 1 \pmod{p_i^{a_i}} \quad (i = 1, \cdots, s).$$

Then (15) holds for

$$\psi_0 = \sum_{i=1}^{s}\left(\frac{m}{p_i^{a_i}}\right)^{\phi(p_i^{a_i})} x_i, \qquad \psi_1 = \sum_{i=1}^{s}\left(\frac{m}{p_i^{a_i}}\right)^{\phi(p_i^{a_i})} y_i.$$

Theorem XII. *The unit residues modulo p^a are*

$$U + \sum_{c=1}^{a-1} p^c \sum_{k=1}^{e_c} f_{ck}\, u^{e_c - k}, \tag{16}$$

where U ranges over the unit residues modulo p and $f = 0, 1, \cdots, p-1$.

For, by (9) and (14), this set gives a complete set of residues modulo p^a which are prime to $F(u)$. If V is one of them,

$$V\psi_0 - F(u)\,\psi_1 \equiv 1, \qquad V\psi_0 \equiv 1 \pmod{p^a}.$$

The remaining residues modulo p^a are divisible modulo p by a factor of $F(u)$. If such a residue N is a unit residue, modulo p^a,

$$NZ \equiv 1 \pmod{p^a}, \qquad N \equiv N' g_i \pmod{p}.$$

Hence

$$N' g_i Z \equiv 1, \qquad F(u)/g_i \equiv 0 \pmod{p}.$$

Theorem XIII. *The unit elements of the algebra are*

$$U_{p^a}\left(\frac{m}{p^a}\right)^{\phi(p^a)} + U_{q^\beta}\left(\frac{m}{q^\beta}\right)^{\phi(q^\beta)} + \cdots + U_{r^\gamma}\left(\frac{m}{r^\gamma}\right)^{\phi(r^\gamma)}, \tag{17}$$

where U_{p^a} ranges over the unit residues modulo p^a, etc.

When an element U given by (17) is considered as a function of u, it is prime to $F(u)$ modulo p, to $F_q(u)$ modulo q, etc., where

$$F_q(u) \equiv 0 \pmod{q^\beta}$$

is the congruence of smallest degree satisfied by u modulo q^β. Now

$$M = \left(\frac{m}{p^a}\right)F(u) + \left(\frac{m}{q^\beta}\right)F_q(u) + \cdots + \left(\frac{m}{r^\gamma}\right)F_r(u)$$

is prime to U with respect to any prime factor of m as modulus. By the lemma, we may set

$$U\psi_0 - M\psi_1 \equiv 1 \pmod{m}.$$

Thus $U\psi_0 = 1$ in the algebra, and U is a unit.

For an element N not of type (17), there is a prime factor μ of m and a factor G of $F_\mu(u)$ modulo μ such that

$$N \equiv GN' \pmod{\mu}.$$

Then N is not a unit. For if so we would have

$$GN'K \equiv 1, \qquad F_\mu(u)/G \equiv 0 \pmod{\mu}.$$

By Theorems XI and XIII, the number of unit residues modulo p^a is

$$V(p^a) = p^E \prod_{i=1}^{s}\left(1 - \frac{1}{p^{n_i}}\right), \qquad E = e_0 + e_1 + \cdots + e_{a-1}. \tag{18}$$

Hence by (17) the number of units in the algebra is

$$V(p^\alpha) \cdot V(q^\beta) \cdots V(r^\gamma).$$

9. Write relation (7) in the form

$$p^k F_k(u) \equiv 0 \pmod{p^\alpha} \qquad (k = 0, 1, \cdots, \alpha - 1). \tag{19}$$

Let R be the polynomial in u of degree $< e_i$ for which

$$F_{i-1} = F_i \psi_i + R.$$

Then $p^i R \equiv 0 \pmod{p^\alpha}$. Hence, by the definition of e_i, each coefficient in $R(u)$ is divisible by p. Thus

$$F_{i-1} = F_i \psi_i + pR_i \qquad (i = 1, \cdots, \alpha - 1). \tag{20}$$

It follows at once that

$$F_0 \equiv F_{\alpha-1} \psi_1 \psi_2 \cdots \psi_{\alpha-1} \pmod{p}.$$

Now $e_{i-1} \geqq e_i$. Unless $e_{i-1} = e_i$, and therefore $F_{i-1} = F_i$, ψ_i is not independent of u. *Hence F_0 has at least as many factors modulo p as there are distinct integers in the set $e_0, e_1, \ldots, e_{\alpha-1}$.*

HENSEL* has given a reduction of the modular system

$$(p^\alpha, f_1(x), f_2(x), \cdots, f_u(x)),$$

in which p is the least integer dividing the system, to a canonical form

$$[p^\alpha, p^{\alpha-1} f_1'(x), p^{\alpha-2} f_2'(x), \cdots, f_\alpha'(x)],$$

in which the f_i' are connected by certain relations and the coefficient of the highest power of x in each f_i' is prime to p. The classes of residues with respect to this modular system define a monomorphic finite algebra. The f_i' have the same properties as the functions F_i in the abstract algebra with the base $m = p^\alpha$. Under the latter restriction, F_0, $pF_1, \ldots, p^{\alpha-1} F_{\alpha-1}$ are zero in the algebra.

* KRONECKER'S *Vorlesungen über Zahlentheorie*, vol. 1, pp. 202–211.

ON THE DEGREE OF CONVERGENCE OF THE DEVELOPMENT OF A CONTINUOUS FUNCTION ACCORDING TO LEGENDRE'S POLYNOMIALS*

BY

DUNHAM JACKSON

It is the purpose of this paper to demonstrate three theorems and two corollaries, whose formal statement will be preceded by a brief explanation. The treatment is suggested by recent papers of LEBESGUE,† in which corresponding problems are discussed for the case of trigonometric series. A part of the work of LEBESGUE is reproduced at length in my thesis,‡ and where the developments of the present paper are closely parallel to those in the trigonometric case I shall occasionally refer the reader to one of those other presentations.

I.

Let $\omega(\delta)$ be a positive function of the positive variable δ, which has the property that as δ increases, $\omega(\delta)$ never decreases, and $\omega(\delta)/\delta$ never increases, while $\lim_{\delta=0}\omega(\delta)=0$. Let $f(x)$ be a function of x, defined and continuous in the interval $-1 \leqq x \leqq 1$. Let $r_n(x)$ denote the difference

$$\sum_{m=0}^{n} a_m P_m(x) - f(x),$$

where the functions $P_m(x)$ are the polynomials of LEGENDRE, and

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\,P_m(x)\,dx.$$

* Presented to the Society, February 24, 1912.

† LEBESGUE, Annales de la Faculté de Toulouse, series 3, vol. 1 (1909), pp. 25–117. Referred to hereafter as LEBESGUE I.

—— Bulletin de la Soc. Math. de France, vol. 38 (1910), pp. 184–210. Referred to as LEBESGUE II.

‡ *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Dissertation, Göttingen, 1911. Referred to as Thesis.

It will be convenient to use a notation of LANDAU* expressing the asymptotic relation of two functions $g_1(n)$, $g_2(n)$, of a variable n, as n becomes infinite, $g_2(n)$ being supposed positive for all values of n from some point on. The expression $g_1(n) = O(g_2(n))$ means that the ratio $g_1(n)/g_2(n)$ remains finite as n becomes infinite, and the expression $g_1(n) = o(g_2(n))$ means that $\lim_{n=\infty} g_1(n)/g_2(n) = 0$.

Then the following is true for any value of x *interior* to the interval $(-1, 1)$, and uniformly throughout any closed interval interior to that interval:

THEOREM I. *If* $f(x)$ *satisfies the condition that*

$$|f(x_2) - f(x_1)| \leqq \omega(\delta)$$

whenever x_1 *and* x_2 *are any two points of the closed interval* $(-1, 1)$ *such that*

$$|x_2 - x_1| \leqq \delta,$$

then†

$$|r_n(x)| = O\left(\omega\left(\frac{1}{n}\right)\log n\right).$$

A special case of this is the

COROLLARY. *If* $\lim_{\delta=0} \omega(\delta)\log\delta = 0$, *the development of* $f(x)$ *according to Legendre's polynomials is convergent.*‡

This statement also holds for any interior point of the interval $(-1, 1)$ and uniformly in any closed interval interior to that interval. As to the end-points $+1$, -1, no conclusion is drawn.

To prove Theorem I, I shall make use of a proposition established in my thesis,§ which, when $\omega(\delta)$ is restricted as above, amounts to a statement that $f(x)$ can be approximately represented throughout the interval by a polynomial of degree n or lower, with a maximum error not exceeding a constant multiple of $\omega(1/n)$, for all values of n. This being granted, a process of

* LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig and Berlin, 1909, pp. 31, 59, 61. LEBESGUE uses occasionally a notation which looks similar to this, with quite a different meaning.

† Cf. LEBESGUE II, pp. 199–202 of the volume, 16–19 of the article. By saying that such a relation as this "holds uniformly in an interval," I mean, of course, that the ratio of the left-hand to the right-hand member, regarded as a function of two variables, remains finite for all values of n from a certain point on and all values of x in the interval.

‡ A somewhat similar condition for convergence, which however does not include the present one and is not included in it, is given by DINI, *Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*, Pisa, 1880. See pp. 281–282, § 116; in § 72, to which the reader is referred back, see especially condition (*b*).

§ Thesis, Satz V, p. 40. The polynomials there called $P_n(x)$ have nothing to do with the polynomials of LEGENDRE. A simplified proof of Satz IV of the thesis, which is used in establishing Satz V, is given in another article by the author presently to appear in these Transactions, *On approximation by trigonometric sums and polynomials*. Another interesting special case of the present theorem is that of the LIPSCHITZ condition, $\omega(\delta)$ = constant $\times \delta$.

reasoning which is readily transferred from the trigonometric case to the present one reduces our problem to that of showing that the absolute value of the remainder $r_n(x)$, formed for any continuous function $f(x)$ whose absolute value does not exceed 1 in the interval $-1 \leqq x \leqq 1$, can never be greater than a constant multiple of $\log n$, as long as x is restricted to an interval $(-1+\epsilon,\ 1-\epsilon)$, $\epsilon > 0$. The essential argument leading to this simplification of the problem is that the nth partial sum of the development of a polynomial of the nth degree according to the functions $P_m(x)$ is identical with the polynomial itself. More simply still, it is seen by writing down at length the expression for this partial sum that it is sufficient to show that, if

$$I_n(x) = \int_{-1}^{1} |K_n(x, t)|\, dt,$$

where

$$K_n(x, t) = \sum_{m=0}^{n} \frac{2m+1}{2} P_m(x) P_m(t),$$

then

$$I_n(x) = O(\log n)$$

uniformly in any interval $(-1+\epsilon,\ 1-\epsilon)$, $\epsilon > 0$.* The notation K_n is adopted from a recent paper of HAAR,† to the first pages of which I shall have occasion to refer.

From the recursion formula for $P_n(x)$ follows readily the identity‡

$$(1) \qquad K_n(x, t) = \frac{n+1}{2} \frac{P_{n+1}(x) P_n(t) - P_n(x) P_{n+1}(t)}{x - t}.$$

We shall use also the asymptotic formula

$$(2) \qquad P_n(x) = \sqrt{\frac{2}{n\pi \sin\theta}} \left[\cos\left((n+\tfrac{1}{2})\theta - \frac{\pi}{4} \right) + \frac{\rho_n(\theta)}{n} \right],$$

where x is set equal to $\cos\theta$, $\pi \geqq \theta \geqq 0$, and $\rho_n(\theta)$ is a function which remains below a certain constant in absolute value for all values of n, and all values of x in the interval $(-1+\frac{1}{2}\epsilon,\ 1-\frac{1}{2}\epsilon)$; it is assumed that ϵ is held fast. It follows at once that in this interval $|P_n(x)|$ never exceeds a constant multiple of $1/\sqrt{n}$, let us say

$$|P_n(x)| \leqq \frac{H}{\sqrt{n}}.$$

* Cf. LEBESGUE I, pp. 116–117; LEBESGUE II, pp. 196–198, 201, of the volume, 13–15, 18, of the article; Thesis, pp. 49–50.

† HAAR, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 132–142; pp. 132–135.

‡ CHRISTOFFEL, Crelle's Journal, vol. 55 (1858), pp. 61–82; p. 73.

It is further pointed out by HAAR that*

$$\int_{-1}^{1} |P_n(x)|\,dx = O(1/\sqrt{n}).$$

As the proposition to be proved will not be affected by neglecting a finite number of values of n, let us suppose n so large that $1/n < \epsilon/2$. Then the points $x - 1/n$, $x + 1/n$ surely lie inside the interval $(-1 + \frac{1}{2}\epsilon, 1 - \frac{1}{2}\epsilon)$, if, as we shall suppose throughout, x is a point of the interval $(-1 + \epsilon, 1 - \epsilon)$. The integral $I_n(x)$ may be broken up into five parts,

$$\int_{-1}^{1} |K_n(x,t)|\,dt = \int_{-1}^{-1+\epsilon/2} + \int_{-1+\epsilon/2}^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^{1-\epsilon/2} + \int_{1-\epsilon/2}^{1}$$
$$= I_n' + I_n'' + I_n''' + I_n^{(IV)} + I_n^{(V)}.$$

In I_n' and $I_n^{(V)}$, t takes on only such values that $|x - t| \geqq \frac{1}{2}\epsilon$, and so, by (1),

$$|K_n(x,t)| \leqq \frac{n+1}{\epsilon}\,|P_{n+1}(x)\,P_n(t) - P_n(x)\,P_{n+1}(t)|.$$

Hence

$$I_n' \leqq \frac{n+1}{\epsilon}\left[|P_{n+1}(x)|\int_{-1}^{-1+\epsilon}|P_n(t)|\,dt + |P_n(x)|\int_{-1}^{-1+\epsilon}|P_{n+1}(t)|\,dt\right].$$

Since $P_n(x) = O(1/\sqrt{n})$ and $P_{n+1}(x) = O(1/\sqrt{n})$, uniformly for $-1 + \epsilon \leqq x \leqq 1 - \epsilon$, and

$$\int_{-1}^{-1+\epsilon}|P_n(t)|\,dt < \int_{-1}^{1}|P_n(t)|\,dt = O(1/\sqrt{n}),$$

and similarly for the integral of $|P_{n+1}(t)|$, it follows that there exists a constant which I_n' never exceeds. The same is true of $I_n^{(V)}$.

Proceeding to the second of the five integrals, we find that

$$I_n'' \leqq \frac{n+1}{2}\left[|P_{n+1}(x)|\int_{-1+\epsilon/2}^{x-1/n}\left|\frac{P_n(t)}{x-t}\right|dt + |P_n(x)|\int_{-1+\epsilon/2}^{x-1/n}\left|\frac{P_{n+1}(t)}{x-t}\right|dt\right]$$
$$\leqq \frac{n+1}{2}\left[\frac{2H^2}{n}\int_{-1+\epsilon/2}^{x-1/n}\frac{dt}{x-t}\right] = H^2\frac{n+1}{n}\int_{1/n}^{x+1-\epsilon/2}\frac{du}{u}$$
$$< H^2\frac{n+1}{n}\int_{1/n}^{2}\frac{du}{u} = H^2\frac{n+1}{n}(\log 2 + \log n).$$

* Since

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1},$$

and

$$\int_{-1}^{1}|P_n(x)|\,dx \leqq \sqrt{2\int_{-1}^{1}[P_n(x)]^2\,dx},$$

by SCHWARZ'S inequality.

This remains inferior to a constant multiple of $\log n$. Similar reasoning applies to $I_n^{(IV)}$.

There remains only the third of the integrals. To handle this, we abandon the formula (1) and write $K_n(x,t)$ at length,

$$|K_n(x,t)| = \left|\sum_{m=0}^{n} \frac{2m+1}{2} P_m(x) P_m(t)\right| \leqq \sum_{m=0}^{n} \left|\frac{2m+1}{2} P_m(x) P_m(t)\right|.$$

In any one term of this sum of absolute values, $|P_m(x)| \leqq H/\sqrt{m}$, and, for values of t in the interval of integration, $|P_m(t)| \leqq H/\sqrt{m}$, and so the terms individually remain inferior to a suitably chosen constant.* The sum of the $n+1$ terms does not exceed a constant multiple of n. Therefore the integral of the sum, over an interval of length $2/n$, remains inferior to a constant.

Putting these results together, we see that it is in fact true that $I_n(x) = O(\log n)$, uniformly in the interval $(-1+\epsilon, 1-\epsilon)$.† This result, combined with Theorem V of my thesis, as has been indicated above, establishes Theorem I of the present paper.

Another proposition demonstrated in my thesis‡ is that if $f(x)$ is a function possessing a $(k-1)$th derivative $f^{(k-1)}(x)$ which satisfies a Lipschitz condition,

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda |x_2 - x_1|,$$

in some interval, λ being a constant, then $f(x)$ may be approximately represented in that interval by a polynomial of the nth degree or lower, for all values of n, with a maximum error not exceeding a constant multiple of $1/n^k$. This may be combined with the relation $I_n(x) = O(\log n)$ to establish the correctness of the following statement, which applies to any point, or uniformly to any closed interval, interior to $(-1, 1)$:

THEOREM II. *If $f(x)$ is a function possessing a $(k-1)$th derivative which satisfies a Lipschitz condition throughout the closed interval $(-1, 1)$, then, for this function $f(x)$,*

$$|r_n(x)| = O\left(\frac{\log n}{n^k}\right).$$

A precisely analogous result for the case of FOURIER'S series§ may be de-

* This last statement is of course true also of the term $\frac{1}{2}P_0(x)P_0(t)$, to which the previous reasoning, with the asymptotic formula, does not apply.

† The formal work above is a modification of that of HAAR in the article cited; but he does not have occasion to obtain the relation $I_n(x) = O(\log n)$.

‡ Thesis, Satz II, p. 23.

§ For the case $k = 1$, this theorem is to be found in the article referred to as LEBESGUE II, pp. 199–201 of the volume, 16–18 of the article, as a part of the analogon of Theorem I above; a result closely parallel to, but neither including nor included in, that for the case $k = 2$ of

duced immediately from Satz VII (page 46) and Hilfssatz II (page 49) of my thesis.

II.

The following theorem is in a way complementary to the first; it shows that our first theorem proves as much as can be proved without making other hypotheses than those we have employed. Let $\omega(\delta)$, $f(x)$, $r_n(x)$ have the meaning given them at the beginning of I.

THEOREM III. *There exists a function $f(x)$ which satisfies the condition that*

$$|f(x_2) - f(x_1)| \leqq \omega(\delta)$$

whenever x_1 and x_2 are any two points of the closed interval $(-1, 1)$ such that

$$|x_2 - x_1| \leqq \delta,$$

*while there is an interior point of the interval at which**

$$r_n(x) \neq o(\omega(1/n) \log n).$$

For $\omega(\delta) = 1/|\log \delta|$ this reduces to the

COROLLARY. *There exists a function $f(x)$ for which $\omega(\delta) \log \delta$ remains finite as δ approaches zero,† while the expansion of $f(x)$ according to Legendre's polynomials diverges at an inner point of the interval $(-1, 1)$.*

(It can not, however, diverge to infinity, by virtue of Theorem I.)

We are to construct a function $f(x)$, satisfying the condition stated in the theorem, for which we can show that for an interior point of the interval $(-1, 1)$, and for infinitely many values of the integral index n, $r_n(x)$ is in absolute value greater than $\omega(1/n) \log n$ multiplied by a constant independent of n. We may confine our attention entirely to one point x, and we may leave out of consideration as many values of n as we please, finite or infinite in number, provided only that we still have infinitely many values left. Let us restrict ourselves to the point $x = 0$, and to values of n which are odd and of the form $4n' + 3$. A number of other conditions will successively be imposed on n, but they will all be satisfied by choosing n sufficiently large, and so will be consistent with each other and with the one already introduced.

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the present theorem is stated in an article by WEYL, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 118–131; p. 128, footnote. See also the author's article *On approximation*, etc., referred to in a previous footnote.

* Cf. LEBESGUE II, pp. 202–206 of the volume, 19–23 of the article.

† Of course $f(x)$ will then be continuous.

For any function $f(x)$, keeping the notation previously used,

$$\sum_{m=0}^{n} a_m P_m(x) = \int_{-1}^{1} f(t) K_n(x, t)\, dt.$$

Since $P_n(0) = 0$ when n is odd, we see from (1) that, for n odd,

$$K_n(0, t) = -\frac{n+1}{2} P_{n+1}(0) \frac{P_n(t)}{t};$$

accordingly

$$\sum_{m=0}^{n} a_m P_m(0) = -\frac{n+1}{2} P_{n+1}(0) \int_{-1}^{1} \frac{f(t) P_n(t)}{t} dt. \tag{3}$$

Let us set

$$x = \cos\theta, \qquad \pi \geqq \theta \geqq 0,$$

as was done before in formula (2), and let g be a constant, which will surely exist, such that $|\rho_n(\theta)| \leqq g$ for all values of n and for $\frac{1}{6}\pi \leqq \theta \leqq \frac{5}{6}\pi$.

Since we suppose $n+1$ divisible by 4, it follows that $\frac{1}{2}(n+1)\pi$ is a multiple of 2π, and

$$\begin{aligned}\cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] &= \cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4} - (n+1)\frac{\pi}{2}\right] \\ &= \cos\left[\left(n+\frac{1}{2}\right)\left(\theta - \frac{\pi}{2}\right) - \frac{\pi}{2}\right] \\ &= \sin\left(n+\frac{1}{2}\right)\left(\theta - \frac{\pi}{2}\right),\end{aligned}$$

or, in terms of a new angle

$$\varphi = \frac{\theta}{2} - \frac{\pi}{4}, \qquad \frac{\pi}{4} \geqq \varphi \geqq -\frac{\pi}{4},$$

$$\cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] = \sin(2n+1)\varphi.$$

Let β be a number satisfying the inequalities $-\frac{1}{2} < \beta < 0$; β may be any such number, but when once chosen is to be held fast. If $x = \beta$, then $\theta = \cos^{-1}\beta$ lies between $\frac{2}{3}\pi$ and $\frac{1}{2}\pi$, and $\varphi = \frac{1}{2}\cos^{-1}\beta - \frac{1}{4}\pi$ lies between $\pi/12$ and 0. Let this value of φ be denoted by α.

Let n be so large that $3\pi/(2n+1) \leqq \alpha$, and let p be the largest integer for which $p\pi/(2n+1) \leqq \alpha$, accordingly $p \geqq 3$. By making n sufficiently large, and hence $g/n \geqq |\rho_n(\theta)|/n$ sufficiently small, we can make the roots of $P_n(x)$ in the interval $\frac{5}{6}\pi \geqq \theta \geqq \frac{1}{6}\pi$, $-\frac{1}{2}\sqrt{3} \leqq x \leqq \frac{1}{2}\sqrt{3}$, correspond so closely to roots of $\sin(2n+1)\varphi$ that $P_n(x)$ vanishes for a value of φ,

say $\varphi = \sigma$, which is nearer to $\pi/(2n+1)$ than to any other root of $\sin(2n+1)\varphi$, and for a value $\varphi = \tau$ nearer to $(p+1)\pi/(2n+1)$ than to any other root of $\sin(2n+1)\varphi$, and similarly for each of the intervening roots of $\sin(2n+1)\varphi$. Let the values of x corresponding to σ and τ be denoted by γ and δ respectively. The latter numbers will surely be interior to the interval $(-\frac{1}{2}, 0)$, for large values of n, and we will assume n taken so large that this is the case. Furthermore, $\frac{1}{2}\pi/(2n+1) < \sigma < 2\pi/(2n+1)$, and $\tau > p\pi/(2n+1) > \sigma$.

We are now ready to define an auxiliary function $\chi_n(x, \beta)$, analogous to the function $\chi_n(x, \alpha)$ defined on p. 58 of my thesis, following the example of Lebesgue. Let

$$\chi_n(x, \beta) = 0, \qquad -1 \leqq x \leqq \delta;$$

$$\chi_n(x, \beta) = \frac{1}{\sqrt{n}} P_n(x), \qquad \delta \leqq x \leqq \gamma;$$

$$\chi_n(x, \beta) = 0, \qquad \gamma \leqq x \leqq 1.$$

This is surely a continuous function of x. The formula

$$(x^2 - 1)\frac{d}{dx} P_n(x) = nxP_n(x) - nP_{n-1}(x),$$

together with the fact that, for the values of x that come into consideration, $P_n(x)$ never surpasses a constant multiple of $1/\sqrt{n}$, shows that $\chi_n(x, \beta)$ satisfies a Lipschitz condition with respect to x,

$$|\chi_n(x_2, \beta) - \chi_n(x_1, \beta)| \leqq \Lambda |x_2 - x_1|, \tag{4}$$

where, as long as β lies in the interval $(-\frac{1}{2}, 1)$, Λ is a constant *independent of n and β* as well as of x_1 and x_2.

If we denote now by a_m the coefficient of $P_m(x)$ in the expansion of $\chi_n(x, \beta)$, and by $r_n(x)$ the corresponding remainder

$$\sum_{m=0}^{n} a_m P_m(x) - \chi_n(x, \beta)$$

(the index of r_n is intentionally set equal to that of χ_n), then we have by (3), noticing that $\chi_n(0, \beta) = 0$,

$$|r_n(0)| = \left| \frac{n+1}{2} \frac{P_{n+1}(0)}{\sqrt{n}} \int_{\delta}^{\gamma} \frac{[P_n(t)]^2}{t} dt \right|.$$

Now let $t = \cos\theta$, $\varphi = \frac{1}{2}\theta - \frac{1}{4}\pi$, the determination of θ and φ being rendered

unique as before by the conventions $\pi \geqq \theta \geqq 0$, $\frac{1}{4}\pi \geqq \varphi \geqq -\frac{1}{4}\pi$.

$$[P_n(t)]^2 = \frac{2}{n\pi \sin\theta}\left[\sin(2n+1)\varphi + \frac{\rho_n(\theta)}{n}\right]^2 .$$

$$= \frac{2}{n\pi \sin\theta}\left[\sin^2(2n+1)\varphi + \frac{\bar{\rho}_n(\theta)}{n}\right],$$

where $\bar{\rho}_n$, like ρ_n, is a function which remains finite for all values of n and θ in question.

$$t = \cos\theta = \cos\left[\frac{\pi}{2} + 2\varphi\right] = -\sin 2\varphi, \qquad dt = -\sin\theta d\theta = -2\sin\theta d\varphi.$$

$$\int_\delta^\gamma \frac{[P_n(t)]^2}{t}dt = -\int_\sigma^\tau \frac{2}{n\pi\sin\theta}\cdot\frac{\sin^2(2n+1)\varphi}{\sin 2\varphi}\cdot 2\sin\theta d\varphi$$

$$-\int_\sigma^\tau \frac{2}{n\pi\sin\theta}\cdot\frac{\bar{\rho}_n(\theta)}{n}\cdot\frac{1}{\sin 2\varphi}\cdot 2\sin\theta d\varphi.$$

To form an idea of the magnitude of the latter integral, let us take into account that in the integrand, $\varphi \geqq \sigma$, $2\varphi \geqq 2\sigma > \pi/(2n+1)$, from which it follows that the greatest value attained by $1/\sin 2\varphi$ within the limits of integration is in order of magnitude $O(n)$. As $\sin\theta$ cancels from numerator and denominator, we see that

$$\int_\sigma^\tau \frac{2}{n\pi\sin\theta}\cdot\frac{\bar{\rho}_n(\theta)}{n}\cdot\frac{1}{\sin 2\varphi}\cdot 2\sin\theta d\varphi = O\left(\frac{1}{n}\right),$$

and

$$\int_\delta^\gamma \frac{[P_n(t)]^2}{t}dt = -\frac{4}{n\pi}\int_\sigma^\tau \frac{\sin^2(2n+1)\varphi}{\sin 2\varphi}d\varphi - O\left(\frac{1}{n}\right).$$

Consider now the integral

$$\int_\sigma^\tau \frac{\sin^2(2n+1)\varphi}{\sin 2\varphi}d\varphi.$$

Since within the limits of integration $0 < \varphi < \pi/12$, the integrand is always positive; furthermore $\sin 2\varphi = 2\sin\varphi\cos\varphi < 2\sin\varphi$; and $\sigma < 2\pi/(2n+1)$, $\tau > p\pi/(2n+1)$. Hence

$$\int_\sigma^\tau \frac{\sin^2(2n+1)\varphi}{\sin 2\varphi}d\varphi > \int_{2\pi/(2n+1)}^{p\pi/(2n+1)} \frac{\sin^2(2n+1)\varphi}{\sin 2\varphi}d\varphi$$

$$> \tfrac{1}{2}\int_{2\pi/(2n+1)}^{p\pi/(2n+1)} \frac{\sin^2(2n+1)\varphi}{\sin\varphi}d\varphi.$$

The last integral is shown on p. 60 of my thesis to be greater than

$$\tfrac{1}{6}[\log(p+1) - \log 3] = \tfrac{1}{6}\log(p+1)\left[1 - \frac{\log 3}{\log(p+1)}\right].$$

From the way in which p was chosen,

$$\frac{(p+1)\pi}{2n+1} > \alpha, \qquad p+1 > \frac{2\alpha}{\pi}(n+\tfrac{1}{2}),$$

and if n is sufficiently large,

$$\log(p+1) > \log\frac{2\alpha}{\pi} + \log(n+\tfrac{1}{2}) > \tfrac{1}{2}\log n,$$

also

$$1 - \frac{\log 3}{\log(p+1)} > \tfrac{1}{2}.$$

These inequalities being satisfied, we have

$$\int_\sigma^\tau \frac{\sin^2(2n+1)\varphi}{\sin 2\varphi}\, d\varphi > \frac{1}{48}\log n.$$

In the expression for $|r_n(0)|$, $P_{n+1}(0)$ is this only factor for the magnitude of which a lower limit has not been ascertained. Remembering that we have assumed n to be of the form $4n'+3$, we readily find from the asymptotic expression for $P_n(x)$ that

$$P_{n+1}(0) = \sqrt{\frac{2}{(n+1)\pi}}\left[1 + \frac{\rho_{n+1}(\pi/2)}{n}\right],$$

which for sufficiently large values of n is greater than a constant positive multiple of $1/\sqrt{n}$. Putting the various factors together, we see finally that there exists a positive constant l, independent of n (and also of α), such that from a certain point on*

$$|r_n(0)| > l\frac{\log n}{n}. \tag{5}$$

For any one function $\chi_n(x,\beta)$, however, this inequality has been established only for the particular value of n for which the function was formed.

One further observation concerning the functions $\chi_n(x,\beta)$ will be of importance. It is clear from the definition of the functions and the asymptotic expression for $P_n(x)$ that there exists a constant M such that for all values of n, β, and x, satisfying the conditions that we have imposed,

$$\frac{2n+1}{2\pi}\chi_n(x,\beta) < M.$$

* The point at which this inequality begins to be true may to be sure depend on α, but that is immaterial.

Still following the procedure of Lebesgue, we go on to define a single function $f(x)$ which satisfies a relation analogous to (5) for infinitely many values of n.

Let β_1 be a number satisfying the inequalities that have been imposed on β. Let n_1 be a value of n so large that a function $\chi_{n_1}(x, \beta_1)$ can be defined and the subsequent work carried through as above. For a reason which will appear later, we shall also assume explicitly that n_1 is chosen so large that $g / n_1 < 1 / 10$, where g, as before, denotes a constant not less than the greatest value of $|\rho_n(\theta)|$ in the interval $\frac{1}{6}\pi \leqq \theta \leqq \frac{5}{6}\pi$. From now on the number n_1 so chosen shall be held fast. Let β_2 be a number, still negative, but greater than the upper limit of the interval of values of x in which $\chi_{n_1}(x, \beta_1)$ takes on values different from zero. Let n_2 be a value of n greater than n_1, and so large

(*a*) that the function $\chi_{n_2}(x, \beta_2)$ may be defined and used as above;

(*b*) that the interval in which $\chi_{n_2}(x, \beta_2)$ takes on values different from zero does not trespass upon the corresponding interval for $\chi_{n_1}(x, \beta_1)$;

(*c*) that the error of the partial sum, to terms of degree n_1, of the expansion in series of Legendre's polynomials of a function which nowhere exceeds $M\omega[2\pi / (2n_2 + 1)]$ in absolute value can not be greater numerically at any point than

$$\frac{l}{4} \cdot \frac{2n_1 + 1}{2\pi} \cdot \omega\left(\frac{2\pi}{2n_1 + 1}\right) \cdot \frac{\log n_1}{n_1}.$$

The constants M and l have already been defined. It is readily seen that the last requirement can be fulfilled, as well as the others, since n_1 is fixed and $\omega[2\pi / (2n_2 + 1)]$ approaches zero as n_2 becomes infinite. Then let β_3 and n_3 be similarly chosen with respect to β_2 and n_2, and the corresponding function $\chi_{n_3}(x, \beta_3)$ constructed, and so on.

If $v_1, v_2, v_3, \cdots$ are constants, each of which is equal to 0 or 1, the series

$$f(x) = \frac{1}{4\Lambda} \sum_{j=1}^{\infty} v_j \frac{2n_j + 1}{2\pi} \omega\left(\frac{2\pi}{2n_j + 1}\right) \chi_{n_j}(x, \beta_j),$$

where Λ is the constant of the relation (4), converges for all values of x in the interval $-1 \leqq x \leqq 1$, and the function $f(x)$ so defined is continuous throughout the interval. This appears at once from the structure of the series, without the application of any general theorems, since at any one point not more than one term of the series is different from zero.

If i is any positive integer, we may break up $f(x)$ into three parts: first, the sum of the first $i - 1$ terms of the series; second, the ith term; and third, the sum of the terms from the $(i + 1)$th on. Suppose these three parts, which are of course continuous functions, expanded in series of Legendre's poly-

nomials to terms of degree n_i, and let the remainders in these expansions at the point $x = 0$ be denoted by R_1, R_2, R_3, respectively. In order to make our function $f(x)$ completely determinate, we still have to fix the coefficients v_i, about which we have said nothing except that each shall be equal to 0 or 1. We will suppose $v_1 = 1$, and the others chosen successively so that, for each value of i,

$$|R_1 + R_2| > \frac{l}{2} \cdot \frac{1}{4\Lambda} \cdot \frac{2n_i + 1}{2\pi} \cdot \omega\left(\frac{2\pi}{2n_i + 1}\right) \cdot \frac{\log n_i}{n_i}.$$

If $|R_1|$ alone satisfies this inequality, we make $v_i = 0$, and so $R_2 = 0$; if not, we make $v_i = 1$, and then the inequality is surely satisfied, in consequence of (5). The third part of $f(x)$ nowhere exceeds $\frac{1}{4}(M/\Lambda)\,\omega(2\pi/(2n_{i+1}+1))$ in absolute value, and so

$$|R_3| \leqq \frac{l}{4} \cdot \frac{1}{4\Lambda} \cdot \frac{2n_i + 1}{2\pi} \cdot \omega\left(\frac{2\pi}{2n_i + 1}\right) \cdot \frac{\log n_i}{n_i},$$

since we have supposed n_{i+1} chosen so large that this conclusion may be drawn.* Consequently†

$$|R_1 + R_2 + R_3| > \frac{l}{4} \cdot \frac{1}{4\Lambda} \cdot \frac{2n_i + 1}{2\pi} \cdot \omega\left(\frac{2\pi}{2n_i + 1}\right) \cdot \frac{\log n_i}{n_i}$$

$$\geqq \frac{l}{16\Lambda\pi} \cdot \log n_i \cdot \omega\left(\frac{1}{n_i}\right).$$

This inequality, holding for the infinitely many values n_i of n, shows that for the function $f(x)$ the reminder $r_n(0)$ is not $o(\omega(1/n)\log n)$.

It remains to be shown that

$$|f(x_2) - f(x_1)| \leqq \omega(|x_2 - x_1|), \tag{6}$$

where x_1, x_2, are any two different points of the interval $(-1, 1)$.

We may distinguish two possibilities as to the position of these points: both may belong to a single interval throughout which $f(x)$ has a single definition as a constant times one of Legendre's polynomials, or they may not both belong to such an interval. The former case is the only one that requires extended consideration. If we can show that in that case

$$|f(x_2) - f(x_1)| \leqq \tfrac{1}{2}\omega(|x_2 - x_1|), \tag{7}$$

the inequality (6) will hold in all cases.‡

* See (c) above.

† Cf. Thesis, pp. 62–64, 67, 68–69.

‡ Cf. Thesis, p. 68.

Let us suppose, then, that x_1 and x_2 are two points of an interval throughuot which the equation holds:

$$f(x) = \frac{1}{4\Lambda} \cdot \frac{2n+1}{2\pi} \omega\left(\frac{2\pi}{2n+1}\right) \cdot \frac{1}{\sqrt{n}} P_n(x),$$

where n is some fixed integer, all the other terms of the series for $f(x)$ being identically zero in the interval. As will presently appear, it is easy to show that (7) holds provided that the values x_1 and x_2 are sufficiently close to each other. But this in itself is not sufficient for our purpose. In the case of trigonometric series* it was possible to find points $\bar{x}_1$ and $\bar{x}_2$ at which the function in question takes on the same values as at x_1 and x_2 respectively, while $\bar{x}_1$ and $\bar{x}_2$ are so near together that the relation (7) can be obtained at once. In the present case it is no longer obvious that we can find such points, for the arches of the curve $y = f(x)$ in the interval under consideration are now not all of the same height, and a value attained on one arch is not necessarily attained on every other arch. It is possible to modify the reasoning, however, so that it is still effective. In the first place, if the difference $|x_2 - x_1|$ is not greater than $2\pi/(2n+1)$, we set $\bar{x}_1 = x_1$, $\bar{x}_2 = x_2$. In the contrary case, the argument is as follows: We have supposed n_1, and hence each of the other indices n_i, chosen so large that for the values of x with which we are concerned $|\rho_n(\theta)|/n \leqq g/n \leqq 1/10$, and for these values of x it is always true furthermore that $\frac{1}{2} \leqq \sin\theta \leqq 1$. Consequently the absolute value of the factor

$$\sqrt{\frac{1}{\sin\theta}}\left[\cos\left((n+\tfrac{1}{2})\,\theta - \frac{\pi}{4}\right) + \frac{\rho_n(\theta)}{n}\right], \tag{8}$$

which can be taken out of the expression (2) for $P_n(x)$, is never greater than $\sqrt{2}(1 + 1/10) < 1.6$, and is greater than $1 - 1/10 = 0.9$, which is more than half of 1.6, at every maximum or minimum of the cosine. This function (8), which needs only to be multiplied by a constant in order to be identical with $f(x)$ in the interval under consideration, takes on at every point of the interval where the cosine is equal to $+1$ a value greater than half its greatest positive value, and similarly for negative values. If y is any value of $f(x)$ in the interval, $f(x)$ takes on the value $\frac{1}{2}y$ in the smaller interval between a maximum and an adjacent minimum of the cosine. Such an interval is described while θ varies through $\pi/(n+\frac{1}{2}) = 2\pi/(2n+1)$, and $x = \cos\theta$ consequently varies through not more than the same amount, $2\pi/(2n+1)$. That is, returning to the points x_1, x_2, it is possible to find two values $\bar{x}_1$, $\bar{x}_2$, in an interval such as has just been described, so that

$$f(\bar{x}_1) = \tfrac{1}{2} f(x_1), \qquad f(\bar{x}_2) = \tfrac{1}{2} f(x_2).$$

* Cf. Thesis, pp. 67–68.

In any case, then, there will be points $\bar{x}_1, \bar{x}_2$, such that the following inequalities are satisfied:

$$|f(\bar{x}_2) - f(\bar{x}_1)| \geqq \tfrac{1}{2}|f(x_2) - f(x_1)|, \tag{9}$$

$$|\bar{x}_2 - \bar{x}_1| \leqq \frac{2\pi}{2n+1}, \tag{10}$$

$$|\bar{x}_2 - \bar{x}_1| \leqq |x_2 - x_1|. \tag{11}$$

The rest of the argument offers nothing new in comparison with the trigonometric case. We know that

$$|\chi_n(\bar{x}_2, \beta) - \chi_n(\bar{x}_1, \beta)| \leqq \Lambda |\bar{x}_2 - \bar{x}_1|,$$

consequently

$$|f(\bar{x}_2) - f(\bar{x}_1)| \leqq \tfrac{1}{4} \cdot \frac{2n+1}{2\pi} \cdot \omega\left(\frac{2\pi}{2n+1}\right) \cdot |\bar{x}_2 - \bar{x}_1| \leqq \tfrac{1}{4}\omega(|\bar{x}_2 - \bar{x}_1|),$$

the last inequality resulting from (10) and the fact that $\omega(\delta)/\delta$ is a never increasing function. Applying (9) and (11), and the fact that $\omega(\delta)$ never decreases as δ increases, we infer hence that

$$|f(x_2) - f(x_1)| \leqq \tfrac{1}{2}\omega(|\bar{x}_2 - \bar{x}_1|) \leqq \tfrac{1}{2}\omega(|x_2 - x_1|);$$

this completes the proof of Theorem III with its corollary.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

FUNCTIONAL DIFFERENTIAL GEOMETRY*

BY

LOUIS INGOLD

It is well known that a certain analogy exists between ordinary vectors or points and functions of a variable x in an interval $\alpha \leq x \leq \beta$. Such an analogy, for instance, may be observed in certain formulas of integral equations† as well as in recent papers dealing with identities connecting integrals.‡ It is the purpose of this paper to give some details of this correspondence in the case of differential geometry of curves and surfaces.

In the ordinary theory a surface is defined by a vector whose projections on the axes are

$$y_i = f(i;\ u_1,\ u_2) \qquad (i = 1, 2, 3),$$

depending upon a parameter i. We consider here instead n-dimensional spaces defined by a function

$$f(x;\ u_1,\ u_2,\ \cdots,\ u_n) \qquad (\alpha \leqq x \leqq \beta),$$

depending upon a continuous parameter x; i. e., n-dimensional spaces in a space of infinitely many dimensions.

For *curves* in the space of infinitely many dimensions a sequence of directions are obtained which are generalizations of the tangent, principal normal, and binormal of ordinary curves; also a sequence of curvatures which correspond to the usual first curvature and torsion. For spaces of higher dimensions the usual tangent properties of ordinary surfaces are generalized, and formulas analogous to the formulas of the Grassmann theory are obtained which express relations among the tangents to subspaces. In the concluding sections the

* This paper combines two papers: *Curves in a Function Space*, and *Surfaces in a Function Space*, both presented to the Society November 26, 1910.

† This analogy was emphasized by Professor E. H. MOORE in lectures on integral equations at the University of Chicago, 1905–07; see his *Introduction to a Form of General Analysis* in the *New Haven Mathematical Colloquium*, Yale University Press, New Haven, 1910. The author owes to Professor MOORE the suggestion that the ideas there expressed might be extended to differential relations.

‡ RICHARDSON and HURWITZ, *Note on determinants whose terms are certain integrals*, Bulletin of the American Mathematical Society, vol. 16 (1909), pp. 14–19; also CURTISS, *Relations between the Gramian, the Wronskian and a third determinant connected with the problem of linear dependence*. Bulletin of the American Mathematical Society, vol. 17 (1911), pp. 462–467.

two fundamental forms of surface theory are generalized and formulas are derived which are the extensions of the well-known relations of GAUSS and CODAZZI.

It is assumed that the functions defining the spaces have partial derivatives of all orders with respect to the parameters u, and these derivatives as well as the functions themselves are assumed to be continuous functions of x.

PART I. CURVES IN A SPACE OF INFINITELY MANY DIMENSIONS.

§ 1. *The notion of curve and the parameter s.*

As explained in the introduction, a function $f(x; u)$ is considered to be analogous to the defining vector* of a curve in ordinary space. We shall say that this function defines a curve in space of infinitely many dimensions and shall refer to the curve as the curve f. The expression $\mu\partial f / \partial x$, where μ is independent of x, will be called a tangent to f.

If the parameter u is replaced by a function of some other letter s, say $u = u(s), f(x; u)$ is transformed into a new function $\varphi(x; s)$. It is possible to choose the new parameter so that

$$\int_\alpha^\beta \left(\frac{\partial \varphi(x; s)}{\partial s}\right)^2 dx = 1. \dagger \tag{1}$$

For

$$\frac{\partial \varphi(x; s)}{\partial s} = \frac{\partial f(x; u)}{\partial u}\frac{du}{ds},$$

and hence

$$\int_\alpha^\beta \left(\frac{\partial \varphi(x; s)}{\partial s}\right)^2 dx = \left(\frac{du}{ds}\right)^2 \int_\alpha^\beta \left(\frac{\partial f(x; u)}{\partial u}\right)^2 dx.$$

Equation (1) is satisfied if

$$ds = \sqrt{\int_\alpha^\beta \left(\frac{\partial f(x; u)}{\partial u}\right)^2 dx}\, du,$$

from which s may be found in terms of u.

* This part of the present paper contains the essential features of a previous paper entitled *Outline of a vector theory of curves*, which was presented to the Society, November 27, 1909. Since then has appeared a paper by E. RATH, *Die Frenetshen Formeln in R_n*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 19 (1910), pp. 269–272, which also treats the subject of space curves vectorially. See also W. Fr. MEYER, *Ausdehnung der Frenetschen Formeln und Verwandter auf den R_n*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 19 (1910), pp. 160–169; and BRUNEL, *Sur les Propriétés Métriques des Courbes Gauches dans un espace linaire à n dimensions*, Mathematische Annalen, vol. 19 (1882), pp. 37–55.

† The integral of the product of two functions is analogous to the inner product of two vectors, i. e., the product of their lengths into the cosine of the included angle. See KOWALEWSKI, *Einführung in die Determinanten-Theorie*, p. 320 ff.

§ 2. *The sequence of normals and curvatures.*

It will be assumed from this point on that the parameter s is so chosen that equation (1) is satisfied. The derivative

$$\frac{\partial f(x;\, s)}{\partial s}, \tag{2}$$

which is tangent to f, will be denoted by $t(x:s)$.

The following formulas define, in terms of $f(x;\, s)$ and its derivatives with respect to s, a set of functions which we shall call the *normals*. For uniformity $t(x;s)$ is included in the set and is denoted by $n_0(x;s)$. These formulas are

$$\begin{aligned} n_0(x;\, s) &= t(x;\, s), \\ \frac{n_1(x;\, s)}{r_1(s)} &= \frac{\partial t(x;\, s)}{\partial s}, \\ \frac{n_i(x;\, s)}{r_i(s)} &= \frac{\partial n_{i-1}(x;\, s)}{\partial s} + \frac{n_{i-2}(x;\, s)}{r_{i-1}(s)}, \dagger \qquad (i \geqq 2), \end{aligned} \tag{3}$$

where the expressions $1/r_i(s)$ are called the curvatures $k_i(s)$ and are defined by the equations

$$k_1^2(s) = \left(\frac{1}{r_1(s)}\right)^2 = \int_\alpha^\beta \left(\frac{\partial n_0(x;\, s)}{\partial s}\right)^2 dx,$$

$$k_i^2(s) = \left(\frac{1}{r_i(s)}\right)^2 = \int_\alpha^\beta \left(\frac{\partial n_{i-1}(x;\, s)}{\partial s} + \frac{n_{i-2}(x;\, s)}{r_{i-1}(s)}\right)^2 dx.$$

It follows at once that

$$\int_\alpha^\beta n_i^2(x;\, s)\, dx = 1. \tag{4}$$

If the integrand vanishes identically when $i = n$, the integral which defines $k_i(s)$ vanishes and the series of equations (3) terminates. Otherwise there is an infinity of normals.

From the definitions (3) it is possible to write the normals as linear expressions

* This is analogous to the tangent vector of an ordinary curve determined by the set of derivatives $df_i(s)/ds$, where

$$x_1 = f_1(s), \qquad x_2 = f_2(s), \qquad x_3 = f_3(s)$$

are the rectangular coördinates in terms of length of arc at a point on the curve.

† These are the well-known Frenet formulas. They do not appear to have been used previously for the purpose of defining the normals. They have been obtained for a curve in function space by KOWALEWSKI in the paper, *Les Formules de Frenet dans l'espace fonctionnel*, Comptes Rendus, vol. 151 (1910), p. 1338.

in $t(x; s)$ and its derivatives with respect to s. For from (3) by differentiation we find

$$\frac{\partial n_1(x; s)}{\partial s} = \frac{dr_1(s)}{ds}\frac{\partial t(x; s)}{\partial s} + r_1(s)\frac{\partial^2 t(x; s)}{\partial s^2}.$$

After substituting in (3) we have

$$n_2(x; s) = \frac{r_2(s)}{r_1(s)}t + r_2(s)\frac{dr_1(s)}{ds}\frac{\partial t(x; s)}{\partial s} + r_1(s)\,r_2(s)\cdot t(x; s),$$

and continuing in this way we obtain:

$$n_i = A(s)\cdot t + A_1(s)\frac{\partial t}{\partial s} + \cdots + A_{i-1}(s)\frac{\partial^{i-1} t}{\partial s^{i-1}} + r_1(s)\cdot r_2(s)\cdots r_i(s)\frac{\partial^i t}{\partial s^i}, \tag{5}$$

where the coefficients A, $A_1 \cdots A_{i-1}$ are functions of s depending on the r's and their derivatives.

It is easily seen that the first normal $n_1(x; s)$ is orthogonal to the tangent $t(x; s)$. For from the equation

$$\int_\alpha^\beta n_0^2(x; s)\,dx = 1$$

it follows by differentiation that

$$\int_\alpha^\beta n_0\frac{\partial n_0}{\partial s}dx = \frac{1}{r_1(s)}\int_\alpha^\beta n_0 n_1\,dx = 0.$$

Hence n_0 and n_1 are orthogonal.

It will now be shown that all the normals are mutually orthogonal. Assume that $n_0(x; s)$, $n_1(x; s)$, $\cdots$, $n_h(x; s)$ form an orthogonal system. The following considerations show that the system $n_0(x; s)$, $n_1(x; s)$, $\cdots$, $n_h(x; s)$, $n_{h+1}(x; s)$ is also orthogonal. We have immediately the equations:

$$\int_\alpha^\beta n_i n_j\,dx = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \quad (i, j = 0, 1, 2, \cdots, h), \tag{6}$$

and from (6) by differentiation:

$$\int_\alpha^\beta n_i\frac{\partial n_j}{\partial s}dx + \int_\alpha^\beta n_j\frac{\partial n_i}{\partial s}dx = 0 \quad (i, j = 0, 1, 2, \cdots, h). \tag{7}$$

In (7) substitute for $\partial n_i / \partial s$ its value from (3). This gives

$$\int_\alpha^\beta \frac{n_j\, n_{i+1}}{r_{i+1}}\,dx - \int_\alpha^\beta \frac{n_j\, n_{i-1}}{r_i}\,dx + \int_\alpha^\beta n_i \frac{\partial n_j}{\partial s}\,dx = 0 \quad (i, j = 0, 1, 2, \cdots, h).^*$$

Hence by (6)

$$\int_\alpha^\beta n_i \frac{\partial n_j}{\partial s}\,dx = 0 \quad (i = 0, 1, 2, \cdots, h-1;\ j = 0, 1, 2, \cdots, h;\ i + j \pm 1), \tag{8}$$

and by (5) and (6)

$$\int_\alpha^\beta n_{j+1} \frac{\partial n_j}{\partial s}\,dx = \frac{1}{r_{j+1}} \qquad (j = 0, 1, 2, \cdots, h-1), \tag{9}$$

$$\int_\alpha^\beta n_{j-1} \frac{\partial n_j}{\partial s}\,dx = -\frac{1}{r_j} \qquad (j = 0, 1, 2, \cdots, h).$$

By using these results $n_{h+1}(x; s)$ is seen to be orthogonal to all the previous n's; for from (3) by multiplying by n_j and integrating

$$\int_\alpha^\beta \frac{n_{h+1}\, n_j}{r_{h+1}}\,dx = \int_\alpha^\beta \frac{\partial n_h}{\partial s}\, n_j\,dx + \int_\alpha^\beta \frac{n_{h-1}\, n_j}{r_h}\,dx \quad (j = 0, 1, 2, \cdots, h).$$

In this equation if $j < h - 1$ the two integrals on the right vanish by (6) and (8). If $j = h - 1$ they cancel each other by (9) and (4), and if $j = h$ they vanish by (6) and (7). Thus the induction is complete.

§ 3. *On the vanishing of the j-th curvature.*

In the ordinary theory of curves it is shown that in case the second curvature (torsion) vanishes identically the curve is a plane curve. A corresponding result holds for curves in a function space. In order to obtain this result, it is convenient to have another expression for the j-th curvature which we proceed to develop. Let $x, x_1, \cdots, x_j$, be $j + 1$ independent variables on the interval $\alpha \leq x_i \leq \beta$, and consider the following integral:

$$\int_\alpha^\beta \frac{1}{(j+1)!} \begin{vmatrix} n_0(x; s) & n_0(x_1; s) & \cdots & n_0(x_j; s) \\ n_1(x; s) & n_1(x_1; s) & \cdots & n_1(x_j; s) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ n_j(x; s) & n_j(x_1; s) & \cdots & n_j(x_j; s) \end{vmatrix}^2 dx\, dx_1 \cdots dx_j.$$

* If $i = 0$ it is to be understood that the second term of this formula does not appear. This is the same as defining the curvature k_0 to be 0.

For convenience, we write this in the form

$$\int_\alpha^\beta [n_0, n_1, \cdots, n_j]^2 \, dx \, dx_1 \cdots dx_j,$$

where the bracket stands for the above determinant divided by the square root of $(j+1)!$ and is analogous to the Grassmann outer product of $j+1$ vectors.

To determine the value of this integral, consider that the determinant expanded consists of $(j+1)!$ terms which are squares, together with certain cross products. The general squared term contributes the value

$$\frac{1}{(j+1)!}\int_\alpha^\beta n_{m_0}^2(x; s)\, n_{m_1}^2(x_1; s) \cdots n_{m_j}^2(x_j; s)\, dx\, dx_1, \cdots, dx_j = \frac{1}{(j+1)!}$$

since by equation (4)

$$\int_\alpha^\beta n_{m_i}^2(x; s)\, dx = 1.$$

Hence all of these terms together yield unity.

The integral of each cross product is zero since each such term must contain as a factor at least one combination like $n_l(x_i; s) \cdot n_h(x_i; s)$, where h and l are different, and the integral of this product is zero because n_l and n_h are orthogonal. The entire integral, therefore, reduces to unity. The square root of the integral of the square of a function $F(x, x_1, \cdots, x_j)$ taken over the region $\alpha \leq x_i \leq \beta$ is called the *norm* of F with reference to the variables $x, x_1, \cdots, x_j$. We have, therefore,

$$\text{Norm}\,[n_0, n_1, \cdots, n_j] = 1. \tag{10}$$

We may, therefore, write the j-th curvature k_j in the form

$$k_j = \frac{1}{r_j} = \frac{1}{r_j}\,\text{norm}\,[n_0, n_1, \cdots, n_j],$$

and substituting for n_j its value from equation (5), we may write the result as the norm of a linear combination of bracket terms

$$\left[n_0, n_1, \cdots, n_{j-1}, \frac{\partial^i t}{\partial s^i}\right] \qquad (i = 0, 1, 2, \cdots, j).$$

These terms all vanish except the one for which $i = j$. Hence

$$k_j = \frac{r_1 r_2 \cdots r_j}{r_j}\,\text{norm}\left[n_0, n_1, \cdots, n_{j-1}\frac{\partial^j t}{\partial s^j}\right].$$

Substituting in like manner for $n_{j-1}, n_{j-2}, \cdots, n_0$, we have finally

$$k_j = r_1^j\, r_2^{j-1} \cdots r_{j-1}^2\,\text{norm}\left[t, \frac{\partial t}{\partial s}, \frac{\partial^2 t}{\partial s^2}, \cdots, \frac{d^j t}{\partial s^j}\right]. \tag{11}$$

We now prove the following theorem.

The necessary and sufficient condition that there exist a finite series for $t(x; s)$ *of the form*

$$(12) \quad t(x; s) = a_1(s)\, e_1(x) + a_2(s)\, e_2(x) + \cdots + a_j(s)\, e_j(x),$$

is that the j-th curvature be the first of the curvatures to vanish identically.

For, if $k_j = 0$, we have

$$\int_a^\beta \left(\frac{\partial n_j}{\partial s} + \frac{n_{j-2}}{r_{j-1}} \right)^2 dx = 0$$

and hence

$$\frac{\partial n_{j-1}}{\partial s} = -\frac{1}{r_{j-1}} n_{j-2}.$$

If the values of n_{j-2} and $\partial n_{j-1} / \partial s$ obtained from (5) are substituted in this equation, it becomes an ordinary linear differential equation of order j for $t(x; s)$.

Let $a_1(s)$, $a_2(s)$, $\cdots$ $a_j(s)$ be j independent solutions of the differential equation satisfied by $t(x; s)$. Then every solution which is a function of both x and s can be written in the form (12), and hence t can be written in this form. Conversely, if t has the form (12), substitution in (11) shows that $k_j \equiv 0$. The functions $e_1(x)$, $e_2(x)$, $\cdots$, $e_j(x)$ are linearly independent. For if not, suppose that all can be expressed linearly in terms of g of them $(g < j)$. Then, from (12) t can be expressed linearly in terms of g of the e's, and hence by (11)

$$k_g \equiv 0 \ (g < j),$$

but this is contrary to the assumption made above that k_j is the *first* of the curvatures to vanish identically.

Part II. Spaces of n Dimensions in Space of an Infinite Number of Dimensions.

§ 4. *The notion of space vector.*

We go at once from one parameter u to any number n of independent parameters and consider a function $f(x; u_1, u_2, \cdots, u_n)$ of x and n parameters $u_1 \cdots u_n$ Such a function will be said to characterize a space of locus of n dimensions which will be called the space f. The different functions of x obtained from f by giving fixed values to the parameters u, may be regarded as the points of the space in question, or they may also be thought of as vectors from the origin to points of the space f.

It is assumed that no linear relation connects the partial derivatives $\partial f(x; u_1, u_2 \cdots u_n) / \partial u_i$. It follows that the function f, for arbitrary values

of $u_1, u_2, \cdots u_n$, cannot be written as a function of x and fewer parameters. In case only such values of the parameters u are considered as can be expressed in terms of k parameters $v_1, v_2, \cdots, v_k$ $(k < n)$, the function f becomes a function of the form

$$f(x; u_1, u_2, \cdots, u_n) = \varphi(x; v_1, v_2, \cdots, v_k).$$

The function φ then characterizes a space of k dimensions. We shall say that the space φ lies in the space f. In the special case $k = 1$, φ is a *curve lying in* f.

The curves defined by

$$u_1, u_2, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n = \text{const.} \quad (i = 1, 2, \cdots, n).$$

will be called the parametric curves corresponding to the parameters u_i.

The space

$$\varphi = u_1 e_1(x) + u_2 e_2(x) + \cdots + u_n e_n(x),$$

where the u's are independent parameters and the e's are linearly independent functions of x alone, will be called a *linear* space of n dimensions.

It is known from the theory of orthogonal functions that there exists a set of normed and mutually orthogonal functions

$$\epsilon_1(x), \epsilon_2(x), \cdots, \epsilon_n(x)$$

in terms of which the e's can be expressed linearly. We may therefore write

$$e_i(x) = \sum_{j=1}^{n} a_{ij}\, \epsilon_j(x).$$

If these values are substituted in the expression for the linear space above, it becomes

$$\varphi = \sum_{i=1}^{n} u_i \sum_{j=1}^{n} a_{ij}\, \epsilon_j(x) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n'} a_{ij}\, u_i \right) \epsilon_j(x) = \sum_{j=1}^{n} v_j\, \epsilon_j(x),$$

where the coefficients

$$v_j = \sum_{i=1}^{n} a_{ij}\, u_i$$

are independent parameters. Thus the function φ characterizing a linear space can be expressed in terms of a normed orthogonal set of functions obtained from the e's.

From the above definitions it is seen that if

$$f(x; u_1, u_2, \cdots, u_n) = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r \qquad (n < r),$$

where the a's are functions of $u_1, u_2, \cdots, u_n$, and the e's are functions of x alone, the space f lies in a linear space of r dimensions.

§ 5. *The first fundamental form.*

If the total differential of the function $f(x; u_1, u_2, \cdots, u_n)$ with respect to parameters u_i, be squared and integrated with respect to x from α to β there results a quadratic differential form in the variables u_i;

$$E = \int_\alpha^\beta (df)^2\, dx = \int_\alpha^\beta \left(\sum_{i=1}^n \frac{\partial f}{\partial u_i}\, du_i\right)^2 dx = \sum_{i,j=1}^n \left(\int_\alpha^\beta \frac{\partial f}{\partial u_i}\,\frac{\partial f}{\partial u_j}\, dx\right) du_i\, du_j$$

$$= \sum_{i,j=1}^n E_{ij}\, du_i\, du_j .$$

The form E is called the first fundamental quadratic differential form associated with the function f. It plays the same rôle as the differential form giving length of arc in the ordinary theory. The coefficients E_{ij} are called the first fundamental quantities. The discriminant of E is denoted by

$$\frac{1}{\lambda^2} = |E_{ij}| \qquad (i, j = 1, 2, \cdots, n),$$

and λ always denotes the positive square root.

In the symbolic theory of the invariants of differential forms* E is represented symbolically as the square of a linear form. It has been shown that the expression

$$\int_\alpha^\beta \left(\sum_{i=1}^n \frac{\partial f}{\partial u_i}\, du_i\right)^2 dx$$

may be used as such a symbolic representation of E† and that all of the identities of the symbolic differential invariant theory may be interpreted as identities involving ordinary functions and their integrals.

If $v_1, v_2, \cdots, v_n$ are any n functions of the parameters u_i we may use the notation

$$(v_1, v_2, \cdots, v_n) = \lambda \frac{\partial(v_1, v_2, \cdots, v_n)}{\partial(u_1, u_2, \cdots, u_n)}.$$

In case two sets of functions a, b are used, we may write

$$(a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_{n-k}) = (a, b) = (a, b; k, n-k),$$

with similar notations for the Jacobian of n functions made up of three or more sets.

* See Maschke, *A symbolic treatment of the theory of invariants of quadratic differential quantics in n variables.* Transactions of the American Mathematical Society, vol. 4 (1903), p. 448.

† See the author's paper, *Note on identities connecting certain integrals,* Bulletin of the American Mathematical Society, vol. 17 (1911), pp. 184–189.

§ 6. *Tangents and normals.*

The function $\mu\partial\psi(x;s)/\partial s$, where μ is independent of x, has been called the tangent to the curve $\psi(x;s)$. The tangent to any curve *lying in* a space $f(x;u_1,\cdots,u_n)$ will be called a tangent to f. All of the tangents to f at a given point $\bar{u}$ (i. e., for given values of the parameters, say $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$) lie in the linear space

$$\varphi = v_1\frac{\overline{\partial f}}{\partial u_1} + v_2\frac{\overline{\partial f}}{\partial u_2} + \cdots + v_n\frac{\overline{\partial f}}{\partial u_n},$$

where $\overline{\partial f}/\partial u_i$ represents the value of $\partial f/\partial u_i$ at the point $\bar{u}$. For if the parameters are all functions of s so that

$$f(x;u_1,u_2,\cdots,u_n) = \psi(x;s),$$

then

$$\frac{\partial\psi}{\partial s} = \frac{\partial f}{\partial u_1}\frac{du_1}{ds} + \frac{df}{\partial u_2}\frac{du_2}{ds} + \cdots + \frac{\partial f}{\partial u_n}\frac{du_n}{ds}$$

which is of the form of φ above. Conversely, any linear function of $\partial f/\partial u_1$, $\partial f/\partial u_2, \cdots, \partial f/\partial u_n$ is a tangent to some curve lying in f and is therefore a tangent to f.

If the coefficients v_i in the expression for φ are independent parameters, φ is called the tangent space to the space f at the point $\bar{u}$.

In case f is the linear space

$$f(x;u_1,u_2,\cdots,u_n) = u_1f_1 + u_2f_2 + \cdots + u_nf_n, \tag{13}$$

where $f_1, f_2, \cdots, f_n$ are linearly independent functions of x alone, the tangent space φ coincides with f.

Suppose that we wish to determine the tangents at a given point of a subspace of $n-1$ dimensions determined for the space $f(x;u_1,u_2,\cdots,u_n)$ by an equation of the form

$$v_1(u_1,u_2,\cdots,u_n) = \text{const.}$$

We may select $n-1$ other functions $v_2, \cdots, v_n$ so that the n functions $v_1, v_2, \cdots, v_n$ are independent and transform the coördinates in f from the u's to the v's. Then

$$f(x;u_1,u_2,\cdots,u_n) = \psi(x;v_1,v_2,\cdots,v_n),$$

$$\frac{\partial f}{\partial u_i} = \sum_{j=1}^{n}\frac{\partial\psi}{\partial v_j}\frac{\partial v_j}{\partial u_i} \qquad (i=1,2,\cdots,n),$$

and the solutions of these equations are

$$\frac{\partial\psi}{\partial v_j} = \frac{(v_1,v_2,\cdots,v_{j-1},f,v_{j+1},\cdots,v_n)}{(v_1,v_2,\cdots,v_n)}.$$

Any curve in the subspace of $n-1$ dimensions can be determined by properly selecting $v_2, v_3, \cdots, v_n$ and letting all but one, say v_n, remain constant. The tangent to the curve so determined is $\partial\psi / \partial v_n$, or what is the same thing, the vector

$$\varphi = (w_1, v_2, v_3, \cdots, v_{n-1}, f),$$

where v_1 is replaced by w_1 to show that it is the locus $w_1 = \text{const.}$ which we are studying. The totality of tangents to this locus is found by using arbitrary functions in place of $v_2, \cdots, v_{n-1}$.

By a similar argument it is seen that the totality of tangents to a subspace of $n-k$ dimensions,

$$w_1 = \text{const.}, \quad w_2 = \text{const.}, \quad \cdots, \quad w_k = \text{const.},$$

is given by the formula

$$(w_1, w_2, \cdots, w_k, v_{k+1}, \cdots, v_{n-1}, f), \tag{14}$$

where $v_{k+1}, \cdots, v_{n-1}$ are arbitrary functions.

Consider now the case in which f is a linear space as in equation (13), with the functions $f_1, \cdots, f_n$ normed and orthogonal.

The expression

$$W = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{\partial w}{\partial u_i} \tag{15}$$

is called the normal to the space $w = \text{const.}$ in the space f, since it is orthogonal to every one of the tangents.

To prove this let φ be expanded in the form

$$\varphi = \sum_{i=1}^{n} V_i \frac{\partial f}{\partial u_i},$$

where V_i is the co-factor of $\partial f / du_i$ in the determinant representing φ. Then

$$\int_a^b W\varphi dx = \sum_{i=1}^{n} V_i \frac{\partial w}{\partial u_i} = 0,$$

since the second member is the determinant for φ with f replaced by w.

§ 7. *Auxiliary notions.*

Let $\varphi_1(x), \varphi_2(x), \cdots, \varphi_k(x)$ represent any k functions of x and any other parameters. Then we may use the notation

$$[\varphi_1, \varphi_2, \cdots, \varphi_k] = \frac{1}{\sqrt{k!}} \begin{vmatrix} \varphi_1(x_1) & \varphi_1(x_2) & \cdots & \varphi_1(x_k) \\ \varphi_2(x_1) & \varphi_2(x_2) & \cdots & \varphi_2(x_k) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \varphi_k(x_1) & \varphi_k(x_2) & \cdots & \varphi_k(x_k) \end{vmatrix},$$

where $x_1, x_2, \cdots, x_k$ are independent variables in the region $\alpha \leq x_i \leq \beta$. This expression is analogous to a Grassmann outer product of k vectors, called by him a *simple element* of the k-th order, or of k dimensions. A linear combination of simple elements which cannot be reduced to a single simple element is called a *compound element.*

We shall assume for the present that the functions $f_1(x), f_2(x), \cdots, f_n(x)$ occurring in the function f in equation (13) are normed and mutually orthogonal. The factor λ defined in § 5 is then unity. The determinant $(f(x_1), f(x_2), \cdots, f(x_n))$ defined at the end of § 5 and all its minors are clearly simple elements except for constant factors.

The simple elements corresponding to minors of the k-th order will be called the *fundamental simple elements of the k-th order* with reference to the functions $f_1, f_2, \cdots, f_n$.

The following notations will be found convenient.

$$[f(x), f(y); k, n-k] = \big(f(x_1), \cdots, f(x_k), f(y_1), \cdots, f(y_{n-k})\big),$$

$$[f(x), a; k, n-k] = \big(f(x_1), \cdots, f(x_k), a_1, \cdots, a_{n-k}\big),$$

$$[f(x); k] = [f(x), u_{c'}; k, n-k] = \lambda \frac{\partial\big(f(x_1), f(x_2), \cdots, f(x_k)\big)}{\partial(u_{c_1}, u_{c_2}, \cdots, u_{c_k})}$$

$$= \sqrt{k!}\left[\frac{\partial f(x)}{\partial u_{c_1}}, \frac{\partial f(x)}{\partial u_{c_2}}, \cdots, \frac{\partial f(x)}{\partial u_{c_k}}\right],$$

$$(a_c; k) = (a_c, u_{c'}; k, n-k) = \frac{\partial(a_1, a_2, \cdots, a_k)}{\partial(u_{c_1}, u_{c_2}, \cdots, u_{c_k})}.$$

In these formulas $c = (c_1, c_2, \cdots, c_k)$ represents any combination of k of the integers $1, 2, \cdots, n$, and $c' = (c'_1, c'_2, \cdots, c'_{n-k})$ is the complementary set arranged so that the permutation $(c_1, c_2, \cdots, c_k, c'_1, c'_2, \cdots, c'_{n-k})$ can be obtained from the permutation $(1, 2, \cdots, n)$ by an even number of transpositions.

The elements so defined satisfy the following relations:

$$\int_\alpha^\beta [f_c(x), u_{c'}; k, n-k]^2 \, dx_1 \, dx_2 \cdots dx_k = \frac{1}{k!}, \tag{16}$$

$$\int_\alpha^\beta [f_c(x), u_{c'}; k, n-k][f_d(x), u_{d'}; k, n-k] \, dx_1 \, dx_2 \cdots dx_k = 0, \tag{17}$$

where c and d represent different combinations.* The proof is the same as for formula (10).

* These are special cases of the formula

$$(M) \qquad \int_\alpha^\beta [\varphi_{i_1}(x_1), \varphi_{i_2}(x_2), \cdots, \varphi_{i_k}(x_k)][\psi_1(x_1), \psi_2(x_2), \cdots, \psi_k(x_k)] \, dx_1 \cdots dx_k = \left| \int_\alpha^\beta \varphi_i(x) \psi_j(x) \, dx \right|$$

The normal in the space f to the locus $a_i(u_1, u_2, \cdots, u_n) = \text{const.}$ will be denoted by the corresponding german letter $\mathfrak{a}$. In the element

$$[\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_k]$$

let $\mathfrak{a}_i$ be replaced by its value from formula (15). Then

$$\mathfrak{A}^{(k)} = [\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_k] = \frac{1}{\sqrt{k!}} \sum_c (a_c;\ k)[f_c(x);\ k].$$

This is analogous to the expansion of a k-dimensional vector in terms of the k-dimensional units. But it follows by use of formulas (16) and (17) that

$$\frac{1}{(n-k)!\sqrt{k!}} \int_\alpha^\beta [f(y), f(x); n-k, k][f(y), a; n-k, k]dy_1\, dy_2 \cdots dy_{n-k} = \frac{1}{\sqrt{k!}} \sum_c (a_c;\ k)[f_c(x);\ k].$$

Hence

$$\text{(I)} \quad \mathfrak{A}^{(k)} = \frac{1}{(n-k)!\sqrt{k!}} \int_\alpha^\beta [f(y), f(x); n-k, k] \times [f(y), a;\ n-k, k]dy_1, \cdots, dy_{n-k}.*$$

In particular

$$\text{(II)} \quad \mathfrak{a} = \frac{1}{(n-1)!} \int_\alpha^\beta [f(y), f(x); n-1, 1][f(y), a; n-1, 1]dy_1 \cdots dy_{n-1}.$$

The expression

$$\text{(III)} \quad |\mathfrak{A}^{(k)} = \frac{1}{\sqrt{k!}(n-k)!} \int_\alpha^\beta [f(y), f(x);\ k, n-k] \times [\mathfrak{a}(y_1), \cdots, \mathfrak{a}(y_k)]dy_1 \cdots dy_k$$

is called the complement of $\mathfrak{A}^{(k)}$. If we expand the element

$$[f(y), f(x);\ k, n-k]$$

in terms of the elements $[f_c(x);\ k]$ and write $\mathfrak{A}^{(k)}$ also in terms of them, we

where the determinant on the right is of the kth order, i and j ranging from 1 to k, and where $\varphi_1, \varphi_2, \cdots, \varphi_k$ and $\psi_1 \psi_2, \cdots, \psi_k$ are any functions of x which are such that the products $\varphi_i \psi_j$ are integrable. This result was first published by RICHARDSON and HURWITZ in the paper already referred to, and later by LANDSBERG in the paper, *Theorie der Elementarteiler linearer Integralgleichungen*, Mathematische Annalen, vol. 69 (1910), pp. 227–265. The formula is a special case of a formula proved by Professor MOORE in a course on Determinants at the University of Chicago, summer, 1907.

* The formulas of this and the following section are closely analogous to formulas given by Grassmann. See the *Ausdehnungslehre* (1862), in Grassmann's collected works edited by F. ENGEL.

have by means of (16) and (17)

$$(\text{IV}) \qquad |\mathfrak{A}^{(k)} = \frac{[a, f(x);\ k, n-k]}{\sqrt{(n-k)!}} = \frac{(-1)^{k(n-k)}}{\sqrt{(n-k)!}}[f(x), a;\ n-k, k].$$

Taking complements again, we find

$$(\text{V}) \qquad ||\mathfrak{A}^{(k)} = \frac{(-1)^{k(n-k)}}{(n-k)!\sqrt{k!}}\int_\alpha^\beta [f(y), f(x);\ n-k, k] \times [f(y), a;\ n-k, k]dy_1 \cdots dy_{n-k} = (-1)^{k(n-k)}\mathfrak{A}^{(k)}.*$$

§ 8. *Properties of the elements.*

The three following formulas, (VI), (VII), (VIII), are frequently useful in making reductions:

$$(\text{VI}) \qquad \int_\alpha^\beta \mathfrak{A}^{(k)}\mathfrak{B}^{(k)}dx_1, \cdots, dx_k = \int_\alpha^\beta |\mathfrak{A}^{(k)}|\mathfrak{B}^{(k)}dy_1, \cdots, dy_{n-k}$$
$$= \frac{1}{(n-k)!}\int_\alpha^\beta [f(x), a;\ n-k, k][f(x), b;\ n-k, k]dx_1 \cdots dx_{n-k},$$

$$(\text{VII}) \qquad \frac{1}{(n-k)!}\int_\alpha^\beta [f(x), a;\ n-k, k][f(x), b;\ n-k, k]dx_1 \cdots dx_{n-k}$$
$$= \left|\int_\alpha^\beta [f(x), a_i;\ n-1, 1][f(x), b_j;\ n-1, 1]dx_1 \cdots dx_{n-1}\right|,\dagger \quad (i, j = 1, 2, \cdots, k).$$

$$(\text{VIII}) \qquad \int_\alpha^\beta [f(y), f(x); n-k, k][f(y), a; n-k, k][f(x), b; k, n-k] \times dy_1 \cdots dy_{n-k}dx_1 \cdots dx_k = k!(n-k)!(a_1, \cdots, a_k, b_1, \cdots, b_{n-k}).$$

Formula (VI) is proved by showing each expression to be equal to

$$\sum_c (a_c;\ k)(b_c;\ k).$$

Formula (VII) can be proved by the use of (VI) and formula (M) of the footnote p. (330). For by (VI) the left hand side is the left hand side of (M), and also by (VI) the general term in the determinant on the right is equal to

$$(n-1)!\int_\alpha^\beta \mathfrak{a}_i\mathfrak{b}_j\,dx.$$

Hence the determinant reduces to the right hand side of (M) and the formula is proved. By expanding both sides of formula (VIII) in terms of $(a_c\,;\,k)$,

* See GRASSMAN, l. c., No. 92.

† Ibid., No. 175.

$(b_c; k)$, they are reduced by formulas (16) and (17) to the same expression

$$k!(n-k)!\sum_c (a_c; k)(b_c; k).$$

The preceding formulas enable us to prove that the elements $[f_c(x); k]$ are linearly independent. For suppose there is a linear relation connecting them, say

$$\sum_c p_c[f(x), u_{c'}; k, n-k] = 0,$$

then

$$\sum_c p_c \int_\alpha^\beta [f(x), f(y); k, n-k][f(x), u_{c'}; k, n-k][f(y), u_h; n-k, k]$$

$$\times dx_1 \cdots dx_k dy_1 \cdots dy_{n-k} = \sum_c p_c k!(n-k)!(u_{c_1'}, \cdots, u_{c_{n-k}'}, u_{h_1}, \cdots, u_{h_k}) = 0,$$

by (VIII). But every term of this vanishes except the term for which the combinations c and h are the same, and this reduces to $k!(n-k)!p_h$. Hence $p_h = 0$ for every h.

In the proofs of the remaining formulas of this section, the following theorem on determinants is needed. If $|\alpha_1, \cdots, \alpha_n|$ and $|\beta_1, \cdots, \beta_n|$ denote two determinants of the nth order, then

$$(D) \quad |\alpha_1, \cdots, \alpha_n||\beta_1, \cdots, \beta_n| = \sum_{i=1}^{n} |\beta_i, \alpha_2, \cdots, \alpha_n||\beta_1, \cdots, \beta_{i-1}, \alpha_1, \beta_{i+1}, \cdots, \beta_n|^*.$$

The theorem can be applied to each term of the sum on the right by putting α_2 successively in place of β_1, β_2, etc.; and clearly this process can be repeated until $\alpha_1, \cdots, \alpha_k$ replace in all possible ways a like number of columns of the second determinant.

In this form the theorem will be used to prove the following formula:

$$(IX) \quad \begin{aligned} &\int_\alpha^\beta [f(x), f(y); n-k, k][f(z), a; l, n-l] \\ &\qquad \times [f(y), f(z), b; k, l, r]\, dy_1\, dy_2 \cdots dy_k\, dz_1\, dz_2 \cdots dz_l \\ &= \frac{k!\,l!}{(k+l)!} \sum_c [f(x), a_c; n-k, k] \int_\alpha^\beta [f(y), a_{c'}; n-r, r] \\ &\qquad \times [f(y), b; n-r, r]\, dy_1\, dy_2 \cdots dy_{n-r}, \end{aligned}$$

where $k+l = n-r$ and a_c denotes any combination of k of the a's and a_c' denotes the combination of the remaining r of the a's.† To prove formula (IX)

* See MASCHKE, *Differential parameters of the first order*, these Transactions, vol. 7 (1906), p. 70, equation (1).

† If $r = 0$ there is only one combination a which consists of all of the a's. The right hand side reduces to a single term and the integral factor of this term reduces to $n! = (k+l)!$; whence it is seen that (IX) reduces to (VIII) in this case.

apply the theorem (D) to the first two factors under the integral sign on the left and exchange in all possible ways $f(y_1), \cdots, f(y_k)$ of the first factor for a like number of elements of the second. Adding the results and denoting the left side of (IX) by P, we have

$$P = \sum_h \int_a^\beta [f(x), f(z_t);\ n-k,\ k]\,[f(y), f(z_\nu),\ a;\ k,\ l-k,\ n-l] \times [f(y), f(z), b;\ k,\ l,\ r]\,dy_1 \cdots dy_k\,dz_1 \cdots dz_l$$

$$+ \sum_{h,i} \int_a^\beta [f(x), f(z_h),\ a_i;\ n-k,\ k-1,\ 1] \times [f(y), f(z_{h'}), a_{i'};\ k,\ l-k+1,\ n-l-1] \times [f(y), f(z),\ b;\ k,\ l,\ r]\,dy_1 \cdots dy_k\,dz_1 \cdots dz_l$$

$$+ \sum_{m,j} \int_a^\beta [f(x), f(z_m),\ a_j;\ n-k,\ k-2,\ 2] \times [f(y), f(z_{m'}),\ a_{j'};\ k,\ l-k+2,\ n-l-2] \times [f(y), f(z),\ b;\ k,\ l,\ r]\,dy_1 \cdots dy_k\,dz_1 \cdots dz_l$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+ \sum_s \int_a^\beta [f(x),\ a_s;\ n-k,\ k]\,[f(y), f(x),\ a_{s'};\ k,\ l,\ r] \times [f(y), f(z),\ b;\ k,\ l,\ r]\,dy_1 \cdots dy_k\,dz_1 \cdots dz_l,$$

where $f(z_t)$ is any combination of k of the quantities $f(z_1) \cdots f(z_l)$, and $f(z_\nu)$ is the combination of the remaining ones. This series of sums is for the case in which k is at most equal to l. If $k = l - g$ the first g sums will not appear. Each term of the first sum is equal to P, except perhaps for sign. By applying theorem (D) to the first two factors of each term of any sum except the first, interchanging in all possible ways the $f(x)$ of the first factor for a like number of terms of the second, each sum is reduced to a series of sums of succeeding types, and by successive reductions they are all reduced to a sum of terms of the type appearing in the last sum, each term being multiplied by a numerical factor. Thus we have

$$P = \sum_c K_c\,[f(x),\ a_c:\ n-k,\ k] \int_a^\beta [f(y),\ a_{c'};\ n-r,\ r] \times [f(y),\ b;\ n-r,\ r]\,dy_1 \cdots dy_{n-r},$$

where the coefficients K_c depend only on the integers k, l, n, r, and not at all on the functions a, b.

Now let

$$\{b_1, b_2, \cdots, b_r\} = \{u_{i_1}, u_{i_2}, \cdots, u_{i_r}\} = \{a_{h_1'}, a_{h_2'}, \cdots, a_{h_r'}\},$$

$$\{a_{h_1}, \cdots, a_{h_k}\} = \{u_{g_1}, \cdots, u_{g_k}\}.$$

Then on the right all terms vanish except the one for which the combinations c and h are the same, and the term which remains has the value

$$K_h\,(l+k)!\,[f_{\bar{h}}(x);\ n-k]\,/\sqrt{(n-k)!}.$$

But the expression for P becomes $k!\,l!\,[f_e(x);\ n-k]\,/\sqrt{(n-k)!}$. Since all of the functions $[f_e(x);\ n-k]$ are independent, $[f_e(x);\ n-k]$ and $[f_{\bar{h}}(x);\ n-k]$ are the same and we have finally

$$K_h = \frac{k!\,l!}{(k+l)!}^{*}$$

for every h, and formula (IX) is proved.

From the formulas given, many others can be obtained by multiplying both sides by a factor involving the function f and integrating. For example, taking complements of both sides of (IX), we have

$$\int_\alpha^\beta [f(x), f(y), b;\ k,\ l,\ r]\,[f(y),\ a;\ l,\ n-l]\,dy_1\cdots dy_l$$

$$(X)\qquad = \frac{l!}{(k+l)!\,(n-k)!}\sum_c \int_\alpha^\beta [f(y), a_c, n-r, r]\,[f(y), b; n-r, r]$$

$$\times dy_1\cdots dy_{n-r}\int_\alpha^\beta [f(z), f(x); n-k, k]\,[f(z), a_{c'};\ n-k, k]\,dz_1\cdots dz_{n-k}.\dagger$$

The proofs of formulas in this and the previous sections are for the case in which f is a linear space whose coefficients f_i are normed and orthogonal. The formulas hold, however, when f is general. Formula (VIII)‡ for the special cases $k=1$ and $k=2$ and formula (VII)§ have been proved by Maschke in symbolic notations. His argument applies equally well to functions. The proofs will therefore not be repeated here.

Formula (VIII) can be shown by induction to hold for any value of k. Denoting the left member by M_k and using the determinant theorem (D),

* This represents the number of terms of the type of those on the right side of (IX) which for a given h are found in the form to which P reduces by the above process. The same expression could be obtained for P *for a general space* f and the same reductions would lead to the same number of terms of the above type. Hence K has the same value for the general case.

† See GRASSMANN, l. c., No. 173.

‡ MASCHKE, *A Symbolic Treatment* etc., loc. cit., formulas (34) and (39).

§ MASCHKE, *Differential parameters of the first order*, these Transactions, vol. 7 (1906), p. 74, equation (8).

we find

$$M_k = \int_\alpha^\beta [f(y), f(x), n-k, k]$$
$$\times \left\{ \sum_{i=1}^{n-k} \left(f(y_1), \cdots, f(y_{i-1}), b_1, f(y_{i+1}), \cdots, f(y_{n-k})\right) \right.$$
$$\left. \times \left(f(x_1), \cdots, f(x_k), f(y_i), b_2, \cdots, b_{n-k}\right) \right.$$
$$+ \sum_{i=1}^{k} \left(f(y_1), \cdots, f(y_{n-k}), a_1, \cdots, a_{i-1}, b_1, a_{i+1}, \cdots, a_k\right)$$
$$\left. \times \left(f(x_1), \cdots, f(x_k), a_i, b_2, \cdots, b_{n-k}\right) \right\} dy_1 \cdots dy_{n-k}\, dx_1 \cdots dx_k.$$

By interchanging equivalent variables $x_1 \cdots x_k\, y_1 \cdots y_{n-k}$ each term of the first summation is seen to be equal to $-M_k$. Hence

$$M_k = (n-k)\, M_{k+1} - kM_k,$$

$$M_{k+1} = \frac{k+1}{n-k} M_k = (k+1)\,!\,(n-k-1)!\,(a_1, \cdots, a_k, b_1, \cdots, b_{n-k}).$$

Consequently the formula holds in general, since it holds for $k = 1$ and $k = 2$. It is clear that the same proof is valid in case any of the quantities a_i, b_i are functions of x or an equivalent variable distinct from the variables of integration.

Formula (VIII) may now be used to show the linear independence of the expressions $[f(x), u_c; k, n-k]$ for the general, non-linear, case precisely as at the beginning of this section for a linear space. The proofs given for formulas (IX) and (X) hold for the general case.*

There remain formulas (I), (II), $\cdots$, (VI). Formulas (II) and (III) may be taken as definitions of $\mathfrak{a}$ and $[\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_k]$ respectively for the more general space. Let the expressions for $\mathfrak{a}_1, \cdots, \mathfrak{a}_k$ from (II) be substituted in the determinant $\mathfrak{A}$ in (I). The result is

$$\frac{1}{\sqrt{k!}} \frac{1}{\{(n-1)!\}^k} \left| \int_\alpha^\beta [f(y), f(x_i); n-1, 1] \right.$$
$$\left. \times [f(y), a_j; n-1, 1]\, dy_1 \cdots dy_{n-1} \right|,$$

where the determinant is of the kth order, i and j ranging from 1 to k. This expression by (VII) can be written

$$\frac{1}{\sqrt{k!}} \frac{1}{(n-k)!} \int_\alpha^\beta [f(y), f(x); n-k, k][f(y), a; n-k, k]\, dy_1 \cdots dy_{n-k},$$

* See the footnote to the proof of formula (IX).

the right-hand side of formula (I), which is therefore proved. Using this formula and formula (VIII) with $f(x_1), \cdots, f(x_{n-k})$ in place of $b_1, \cdots, b_{n-k}$, we have

$$|\mathfrak{A}^{(k)} = \frac{1}{\sqrt{k!(n-k)!}} \int_\alpha^\beta [f(y), f(x); k, n-k] \frac{1}{(n-k)!\sqrt{k!}}$$

$$\times \int_\alpha^\beta [f(z), f(y); n-k, k][f(z), a; n-k, k] dz_1 \cdots dz_{n-k} dy_1 \cdots dy_k$$

$$= \frac{1}{k!(n-k)!\sqrt{(n-k)!}} \int_\alpha^\beta [f(z), f(y); n-k, k]$$

$$\times [f(z), a; n-k, k][f(y), f(x); k, n-k] dz_1 \cdots dz_{n-k} dy_1 \cdots dy_k$$

$$= \frac{1}{\sqrt{(n-k)!}} \left(a_1, a_2, \cdots, a_k, f(x_1), f(x_2), \cdots, f(x_{n-k})\right),$$

and this proves formula (IV). In a similar manner formulas (V) and (VI) can be proved.

§ 9. *Functions orthogonal to all tangents of a subspace.*

Given a subspace R_k lying in f, it is possible to find functions tangent to f but orthogonal to all tangents to R_k. In fact, if f is of n dimensions there should be precisely $n-k$ *independent* functions satisfying this condition. These functions are given in the following theorem:

Every function

$$\psi(x) = \int_\alpha^\beta [f(y), f(x), U; k, 1, n-k-1][f(y), a; k, n-k] dy_1 \cdots dy_k,$$

where the U's are arbitrary functions of $u_1, u_2, \cdots, u_n$, is orthogonal to all tangents of the subspace R_k in f defined by the equations

$$a_1 = \text{const.}, \quad a_2 = \text{const.}, \quad \cdots, \quad a_{n-k} = \text{const.}$$

By formula (14), § 6, all tangents to R_k may be written in the form

$$\varphi(x) = [f(x), V, a; 1, k-1, n-k],$$

where the V's are arbitrary functions of $u_1, u_2, \cdots, u_n$. It is to be shown, then, that

$$\int_\alpha^\beta \varphi(x)\psi(x)\, dx = \int_\alpha^\beta [f(y), f(x), U; k, 1, n-k-1][f(y), u; k, n-k]$$

$$\times [f(x), V, u; 1, k-1, n-k]\, dx\, dy_1 \cdots dy_k = 0.$$

By formula (X)

$$\psi(x) = \sum_{i=1}^{n} \int_{\alpha}^{\beta} [f(y), u_i; k+1, n-k-1][f(y), U; k+1, n-k-1]$$

$$\times \int_{\alpha}^{\beta} [f(z), f(x); n-1, 1][f(z), a_i; n-1, 1]\, dy_1 \cdots dy_{k+1} dz_1 \cdots dz_{n-1}.$$

Hence, by (VIII)

$$\int_{\alpha}^{\beta} \varphi(x)\psi(x)\, dx = \sum_{i=1}^{n} M_i \int_{\alpha}^{\beta} [f(z), f(x); n-1, 1][f(z), a_i; n-1, 1]$$

$$\times [f(x), V, a; 1, k-1, n-k]\, dx\, dz_1 \cdots dz_{n-1}$$

$$= (n-1)! \sum_{i=1}^{n} M_i (a_i, V, a; 1, k-1, n-k).$$

But each term of this sum contains a determinant having two columns identical. Hence the sum vanishes, and this proves the theorem.

§ 10. *Normal properties; Christoffel symbols.*

If all of the parameters u_i are functions of the same parameter s, the function f determines a curve $f(x; s)$. The tangent to this curve is given by formula (2) of § 2. Let s be so chosen that the condition (1) holds. Then the first normal n to the curve is defined by (3) to be

$$\frac{1}{r} n = \frac{\partial^2 f(x; s)}{\partial s^2} = \sum_{i,j=1}^{n} \frac{\partial^2 f(x; s)}{\partial u_i \partial u_j} \frac{\partial u_i}{\partial s} \frac{du_j}{ds} + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{d^2 u_i}{ds^2},$$

where $1/r$ is the first curvature and is so chosen that

$$\int_{\alpha}^{\beta} n^2\, dx = 1.$$

Thus *the first normals to all curves lying in f are linearly expressible in terms of the tangents $\partial f / \partial u_i$ and the second partial derivatives $\partial^2 f / \partial u_i \partial u_j$.*

A function which is linearly expressible in terms of the derivatives $\partial f / \partial u_i$, and $\partial^2 f / \partial u_i \partial u_j$, and which is orthogonal to all of the tangents, $\partial f / \partial u_i$, we shall call *a first normal to the space f.*

If the second derivatives $\partial^2 f / \partial u_i \partial u_j$ are linearly independent there are $n(n+1)/2$ linearly independent first normals to f in terms of which all others can be expressed. We write

$$N_{ij}(u_1, u_2, \cdots, u_n) = \frac{\partial^2 f}{\partial u_i \partial u_j} - \sum_{k=1}^{n} \left\{ \begin{matrix} i\ j \\ k \end{matrix} \right\} \frac{\partial f}{\partial u_k}, \tag{18}$$

where the coefficients $\left\{ \begin{matrix} i\ j \\ k \end{matrix} \right\}$ are to be determined as functions of the para-

meters u_i from the condition that the functions N_{ij} are orthogonal to the functions $\partial f / \partial u_i$. To calculate the $\{ {}^{i\ j}_{k} \}$, multiply both sides of equation (18) by $\partial f / \partial u_l$ and integrate, remembering equation (6). This gives

$$\int_a^b \frac{\partial^2 f}{\partial u_i \partial u_j} \frac{\partial f}{\partial u_l} dx = \sum_{k=1}^{n} \left\{ \begin{matrix} i & j \\ k \end{matrix} \right\} E_{kl}.$$

Now the expressions on the left may be shown to be the Christoffel triple index symbols* of the first kind belonging to the quadratic differential form (E), and can be expressed in terms of the functions E_{ij} and their derivatives. Denoting them by $[{}^{i\ j}_{l}]$ we have the system of equations

$$\sum_{k=1}^{n} E_{kl} \left\{ \begin{matrix} i & j \\ k \end{matrix} \right\} = \left[\begin{matrix} i & j \\ l \end{matrix} \right] \qquad (l = 1, 2, \cdots n),$$

from which we obtain

$$\left\{ \begin{matrix} i & j \\ k \end{matrix} \right\} = \lambda^2 \sum_{l=1}^{n} \mathfrak{E}_{kl} \left[\begin{matrix} i & j \\ l \end{matrix} \right],$$

where $\mathfrak{E}_{kl}$ denotes the co-factor of E_{kl} in the determinant $| E_{ij} |$. The expressions $\{ {}^{i\ j}_{k} \}$ are the Christoffel triple index symbols of the second kind belonging to the differential form (E).†

§ 11. *The second fundamental form.*

For the study of properties depending upon the second derivatives of the function f certain other fundamental quantities are important. These may be taken to be the quantities of either of the sets

$$\int_a^b \frac{\partial^2 f}{\partial u_i \partial u_j} \frac{\partial^2 f}{\partial u_r \partial u_s} dx, \qquad \int_a^b N_{ij} N_{rs} dx,$$

or perhaps still others which can be expressed in terms of either of these sets and the first fundamental quantities E. Under certain restrictions the number of these quantities is reduced. Thus in the ordinary theory of spaces of n dimensions lying in a Euclidean space of $n + 1$ dimensions, the number is reduced to $n(n+1)/2$, the fundamental functions being the coefficients of the *second* quadratic differential form. Here, however, for the corresponding developments it is not assumed that f lies in a Euclidean space of $n + 1$ dimensions, but only that all *first* normals to f have the same direction; i. e., that *any two of the normals N_{ij} differ from each other at most by a factor which is independent of x*,‡ so that we may write

$$N_{ij} = L_{ij} N,$$

* See MASCHKE, *A Symbolic Treatment*, etc., loc. cit., p. 455.

† See MASCHKE, *A Symbolic Treatment*, etc., loc. cit., p. 456, equation (67).

‡ The further assumption, that the third and higher derivatives are linearly expressible in terms of the first and second derivatives of f leads to the ordinary case mentioned above.

where L_{ij} is independent of x and where

$$\int_\alpha^\beta N^2 dx = 1.$$

Equation (18) may now be written

$$L_{ij} N = \frac{\partial^2 f}{\partial u_i \partial u_j} - \sum_{k=1}^{n} \left\{ \begin{matrix} i & j \\ k & \end{matrix} \right\} \frac{\partial f}{\partial u_k}. \tag{19}$$

Multiplying both sides of this equation by N and integrating, we have, since N is orthogonal to $\partial f / \partial u_k$,

$$L_{ij} = \int_\alpha^\beta N \frac{\partial^2 f}{\partial u_i \partial u_j} dx.$$

The L_{ij} are the coefficients of a quadratic differential form obtained in the following manner. We have

$$d^2 f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial u_i \partial u_j} du_i du_j + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} d^2 u_i.$$

If we multiply by N and integrate, we find

$$\int_\alpha^\beta (d^2 f) N dx = \sum_{i,j=1}^{n} \left\{ N \frac{\partial^2 f}{\partial u_1 \partial u_j} dx \right\} du_i du_j = \sum_{i,j=1}^{n} L_{ij} du_i du_j.$$

This is called the second fundamental differential form and the coefficients L_{ij} are called the second fundamental quantities associated with the function f.

The following relations are easily obtained by differentiation under the integral sign. From

$$\int_\alpha^\beta N \frac{\partial f}{\partial u_i} dx = 0, \qquad \int_\alpha^\beta N \frac{\partial f}{\partial u_j} dx = 0, \qquad \int_\alpha^\beta N^2 dx = 1$$

we have

$$\int_\alpha^\beta N \frac{\partial^2 f}{\partial u_i \partial u_j} dx = - \int_\alpha^\beta \frac{\partial N}{\partial u_j} \frac{\partial f}{\partial u_i} \partial x = - \int_\alpha^\beta \frac{\partial N}{\partial u_i} \frac{\partial f}{\partial u_j} dx = L_{ij}, \tag{20}$$

$$\int_\alpha^\beta N \frac{\partial N}{\partial u_i} dx = 0. \tag{21}$$

§ 12. *The relations of Gauss and Codazzi.*

We now apply the formulas just obtained in the proof of the extension of the well-known Gauss and Codazzi relations. From equation (19) we have

$$\frac{\partial^2 f}{\partial u_i\,\partial u_j} = \sum_{r=1}^{n} \begin{Bmatrix} i\ j \\ r \end{Bmatrix} \frac{\partial f}{\partial u_r} + L_{ij}\,N,$$

$$\frac{\partial^2 f}{\partial u_i\,\partial u_k} = \sum_{r=1}^{n} \begin{Bmatrix} i\ k \\ r \end{Bmatrix} \frac{\partial f}{\partial u_r} + L_{ik}\,N.$$

Differentiating the first with respect to u_k and the second with respect to u_j and equating the results, we find

$$\frac{\partial^3 f}{\partial u_i\,\partial u_j\,\partial u_k} = \sum_{r=1}^{n} \begin{Bmatrix} i\ j \\ r \end{Bmatrix} \frac{\partial^2 f}{\partial u_r\,\partial u_k} + \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i\ j \\ r \end{Bmatrix}}{\partial u_k} \frac{\partial f}{\partial u_r} + L_{ij} \frac{\partial N}{\partial u_k} + \frac{\partial L_{ij}}{\partial u_k} N$$

$$= \sum_{r=1}^{n} \begin{Bmatrix} i\ k \\ r \end{Bmatrix} \frac{\partial^2 f}{\partial u_r\,\partial u_j} + \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i\ k \\ r \end{Bmatrix}}{\partial u_j} \frac{\partial f}{\partial u_r} + L_{ik} \frac{\partial N}{\partial u_j} + \frac{\partial L_{ik}}{\partial u_j} N.$$

Multiplying in turn by N and $\partial f / \partial u_m$ and integrating, we obtain, by the use of (20) and (21), after transposing,

$$\sum_{r=1}^{n} \begin{Bmatrix} i\ j \\ r \end{Bmatrix} L_{rk} - \sum_{r=1}^{n} \begin{Bmatrix} i\ k \\ r \end{Bmatrix} L_{rj} + \frac{\partial L_{ij}}{\partial u_k} - \frac{\partial L_{ik}}{\partial u_j} = 0, \tag{22}$$

$$\sum_{r=1}^{n} \begin{Bmatrix} i\ j \\ r \end{Bmatrix} \begin{bmatrix} r\ k \\ m \end{bmatrix} - \sum_{r=1}^{n} \begin{Bmatrix} i\ k \\ r \end{Bmatrix} \begin{bmatrix} r\ j \\ m \end{bmatrix} + \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i\ j \\ r \end{Bmatrix}}{\partial u_k} E_{rm}$$

$$- \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i\ k \\ r \end{Bmatrix}}{\partial u_j} E_{rm} + L_{ik}\,L_{jm} - L_{ij}\,L_{km} = 0. \tag{23}$$

The last relation contains the result that the expression $L_{ik}\,L_{jm} - L_{ij}\,L_{km}$ can be expressed in terms of the E_{ij} and their derivatives.

UNIVERSITY OF MISSOURI,
May, 1912.

ON THE EXTENSION OF A THEOREM OF POINCARÉ FOR DIFFERENCE-EQUATIONS*

BY

EDWARD B. VAN VLECK

One of the most useful theorems relating to difference equations is the well-known theorem of POINCARÉ† for the equation

$$(1) \qquad u(n+r) + P_1(n)\,u(n+r-1) + \cdots + P_r(n)\,u(n) = 0,$$

in which the coefficients $P_i(n)$ possess limits A_i for $n = +\infty$. POINCARÉ showed, namely, that if the roots of the equation

$$(2) \qquad y^r + A_1 y^{r-1} + \cdots + A_r = 0$$

are no two of equal modulus, the ratio $u(n)/u(n-1)$ has a limit for $n = +\infty$, and this limit is a root of (2), in general the root of greatest modulus.‡

* Presented to the Society at the Chicago meeting, April 29, 1911.

† American Journal of Mathematics, vol. 7 (1885), p. 213. For careful analysis and further development of the theorem, see PERRON, Crelle's Journal, vol. 136 (1909), p. 17 and vol. 137, p. 6.

‡ In 1904, in an article on continued fractions in these Transactions, p. 253, I gave a proof of POINCARE'S theorem for $r = 2$ which was suitable for application to the class of continued fractions there under consideration. This proof, and consequently its immediate applicability, was not fully understood by PRINGSHEIM and called forth an interesting memoir from him (Sitzungsberichte der Bayerischen Akademie, math.-physikalischer Klasse, 1910, p. 3) devoted in large part to rederiving my results. Later after oral and written communication between us he has recognized the legitimacy of my proof (Ibid., 1911, p. 64). My argument was somewhat condensed, but not more so, it seemed to me, than is current in articles of an investigational character. A few additional lines, which it is worth while to add now, would have removed any difficulty.

When Pringsheim reaches the following theorem (p. 46), he inadvertently fails to attribute it to me, though the attribution is intended in the introduction:

If in the continued fraction

$$A(z) = \frac{1}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots$$

we have $\lim a_n = k$ *for* $n = \infty$, *the continued fraction will converge over the entire plane except* (1) *along the whole or a part of a rectilinear cut drawn from* $z = -\frac{1}{4}k$ *to* $z = \infty$ *with an argument equal to that of the vector from the origin to* $z = -\frac{1}{4}k$, *and except possibly* (2) *at certain isolated points* $p_1, p_2, \cdots$. *Within the plane so cut the continued fraction is holomorphic except at the points* $p_1, p_2, p_3, \cdots$, *which are poles.*

The equation (1) is obviously equivalent to the following linear system of first order:

$$
\begin{aligned}
u_1(n+r) &= -P_1(n)\,u_1(n+r-1) - P_2(n)\,u_2(n+r-1) \\
&\qquad - \cdots - P_r(n)\,u_r(n+r-1), \\
u_2(n+r) &= u_1(n+r-1), \\
u_3(n+r) &= u_2(n+r-1), \\
&\cdots\cdots\cdots\cdots \\
u_r(n+r) &= u_{r-1}(n+r-1).
\end{aligned} \tag{3}
$$

In the following paper an analogue of POINCARÉ'S theorem is sought for the general linear system:

$$(A)\quad y_i(n) = a_{i1}(n)y_1(n-1) + \cdots + a_{ir}(n)y_r(n-1) \quad (i = 1, 2, \cdots, r),$$

in which the coefficients $a_{ij}(n)$ have limiting values A_{ij} for $n = +\infty$. This extension, which is by no means superficially evident, is contained in Theorems 1 and 2 of § 2.

When the coefficients of (A) are constant, it gives for $n = 1, 2, \cdots$ the indefinite iteration of a collineation. This has led me to the following paper on the classification of collineations.

§1. *Preliminary Considerations.*

Consider first the linear substitution (A) for a fixed value of n and seek the linear combinations of $y_1(n-1), \cdots, y_r(n-1)$ which in consequence of the substitution are altered only by a multiplicative constant; i. e.,

$$\sum_{i=1}^{r} c_i\, y_i(n) = \rho \sum_{i=1}^{r} c_i\, y_i(n-1), \tag{4}$$

Inserting here the values of the $y_i(n)$ given in (A), we obtain an identity which gives the following system of equations for the determination of the ratios of the c_i;

$$
\begin{aligned}
&[a_{11}(n) - \rho]\,c_1 + a_{21}(n)\,c_2 + \cdots + a_{r1}(n)\,c_r = 0, \\
&a_{12}(n)\,c_1 + [a_{22}(n) - \rho]\,c_2 + \cdots + a_{r2}(n)\,c_r = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{1r}(n)\,c_1 + a_{2r}(n)\,c_2 + \cdots + [a_{rr}(n) - \rho]\,c_r = 0.
\end{aligned} \tag{5}
$$

The elimination of the c_i gives for ρ the familiar characteristic equation,

$$
\begin{vmatrix}
a_{11}(n) - \rho & a_{21}(n) & \cdots & a_{r1}(n) \\
a_{12}(n) & a_{22}(n) - \rho & \cdots & a_{r2}(n) \\
\cdots & \cdots & \cdots & \cdots \\
a_{1r}(n) & a_{2r}(n) & \cdots & a_{rr}(n) - \rho
\end{vmatrix} = 0. \tag{6}
$$

When n increases indefinitely, every $a_{ij}(n)$ has by hypothesis a limit A_{ij}. The last equation has therefore the limiting form,

$$(7)\qquad \begin{vmatrix} A_{11}-\rho & A_{21} & \cdots & A_{r1} \\ A_{12} & A_{22}-\rho & \cdots & A_{r2} \\ \cdot & \cdot & \cdots & \cdot \\ A_{1r} & A_{2r} & \cdots & A_{rr}-\rho \end{vmatrix} = 0.$$

Suppose now that no two roots R_i of (7) are equal, and let them be arranged in the order of decreasing modulus,

$$(8)\qquad |R_1| \geqq |R_2| \geqq \cdots \geqq |R_r|.$$

Then by taking n sufficiently large the r roots of (6), $\rho_i(n)$ $(i = 1, \cdots, r)$, may be distinguished from one another by virtue of their proximity to the roots R_i of (7). Thus for a sufficiently large value of N we will have

$$|R_i - \rho_i(n)| < \tfrac{1}{2}|R_i - R_j| \qquad (n > N;\ j \neq i).$$

To each $\rho_i(n)$ there corresponds a set of equations (5), which we will now provide with appropriate subscripts and write:

$$(9)\qquad \begin{aligned} &[a_{11}(n) - \rho_i(n)]\,c_{1i}(n) + a_{21}(n)\,c_{2i}(n) + \cdots + a_{r1}(n)\,c_{ri}(n) = 0, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &a_{1r}(n)\,c_{1i}(n) + a_{2r}(n)\,c_{2i}(n) + \cdots + [a_{rr}(n) - \rho_i(n)]\,c_{ri}(n) = 0. \end{aligned}$$

From the theory of linear substitutions it is known that $r-1$ of these equations are independent when the roots of (6) are distinct.* These equations therefore completely determine the ratios of the $c_{ji}(n)$ $(j = 1, \cdots, r)$. For a like reason the limiting equations,

$$(10)\qquad \begin{aligned} &(A_{11} - R_i)\,C_{1i} + A_{21}\,C_{2i} + \cdots + A_{ri}\,C_{ri} = 0, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &A_{1r}\,C_{1i} + A_{2r}\,C_{2i} + \cdots + (A_{rr} - R_i)\,C_{ri} = 0, \end{aligned}$$

determine the ratios of the C_{ji}. Accordingly for each value of i there is some one of the C_{ji} which may be placed equal to 1, and the ratios of the other C_{ji} $(j = 1, \cdots, r)$ to this particular one will be finite. It follows then, since the coefficients of (10) are the limits of those in (9), that these ratios are the limits of the corresponding ratios of the $c_{ji}(n)$ for $n = \infty$. Hence if N is chosen sufficiently large, the $c_{ji}(n)$ which correspond to the particular C_{ji} taken equal to 1, may also be set equal to 1. We suppose hereafter that this has been done. By means of this convention the $c_{ji}(n)$ $(j = 1, \cdots, r)$ will be uniquely determined for $n \geqq N$, and accordingly we shall have r

* See, for example, pages 72–73 of HORN's *Gewöhnliche Differentialgleichungen beliebiger Ordnung.*

definite combinations,

$$v_i(n) = \sum_{j=1}^{r} c_{ji}(n)\, y_j(n-1) \qquad (i = 1, \cdots, r), \tag{11}$$

which possess the following property: When transformed by the linear substitution (A), they undergo the multiplicative transformation

$$\sum_{j=1}^{r} c_{ji}(n)\, y_j(n) = \rho_i(n) \sum_{j=1}^{r} c_{ji}(n)\, y_j(n-1).$$

For the limiting form of (11) we have

$$V_i(n) = \sum_{j=1}^{r} C_{ji}\, Y_j(n-1) \qquad (i = 1, \cdots, r). \tag{12}$$

In the theory of linear substitutions* it is shown that the determinant $|\, c_{ji}(n)\,|$ can not vanish when the roots of (6) are distinct. We may therefore solve (11) for the $y_j(n-1)$ in terms of the $v_i(n)$. Similarly the determinant $|\, C_{ji}\,| \neq 0$, and equations (12) may be solved for the $Y_j(n-1)$. The solutions will be denoted by

$$y_i(n-1) = \sum_{j=1}^{r} d_{ji}(n)\, v_j(n) \qquad (i = 1, \cdots, r), \tag{13}$$

$$Y_i(n-1) = \sum_{j=1}^{r} D_{ji}\, V_j(n), \tag{14}$$

in which D_{ji} is obviously the limit of $d_{ji}(n)$ for $n = \infty$.

The ratios of the D_{ji} can be most easily determined by considering the linear system of difference equations with constant coefficients

$$Y_i(n) = \sum_{j=1}^{r} A_{ij}\, Y_j(n-1) \qquad (i = 1, \cdots, r), \tag{15}$$

to which the limiting relations (12) and (14) may be attached. For this system we have

$$V_i(n+1) = R_i\, V_i(n).$$

Hence

$$Y_i(n) = \sum_{j=1}^{r} D_{ji}\, V_j(n+1) = \sum_{j=1}^{r} D_{ji} R_j\, V_j(n). \tag{16}$$

If the expressions for $Y_i(n-1)$ and $Y_i(n)$ which are given in (14) and (16) are substituted in (15), and the coefficients of each $V_j(n)$ on the two sides of the equation are then equated, we obtain the following r sets of relations

* Cf. HORN, loc. cit., p. 73.

$(j = 1, \cdots, r)$:

$$\begin{aligned}
&(A_{11} - R_j) D_{j1} + A_{12} D_{j2} + \cdots + A_{1r} D_{jr} = 0, \\
&A_{21} D_{j1} + (A_{22} - R_j) D_{j2} + \cdots + A_{2r} D_{jr} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&A_{r1} D_{j1} + A_{r2} D_{j2} + \cdots + (A_{rr} - R_j) D_{jr} = 0.
\end{aligned} \tag{17}$$

This shows that the r coefficients D_{ji} $(i = 1, \cdots, r)$ are the homogeneous coördinates of one of the invariant points of the linear substitution (15).

In considering the effect of the indefinite increase of n upon any solution $y_i(n)$ $(i = 1, \cdots, r)$ of (A), I propose to deal with them through the $v_i(n)$. For this purpose it will be necessary to find the relations subsisting between the $v_i(n+1)$ and $v_i(n)$. With the aid of (A) we obtain

$$\begin{aligned}
v_i(n+1) &= \sum_{j=1}^{r} c_{ji}(n+1)\, y_j(n) = \sum_{j=1}^{r}\sum_{k=1}^{r} c_{ji}(n+1)\, a_{jk}(n)\, y_k(n-1) \\
&= \sum_{j=1}^{r}\sum_{k=1}^{r} c_{ji}(n)\, a_{jk}(n)\, y_k(n-1) + \sum_{j=1}^{r}\sum_{k=1}^{r} [c_{ji}(n+1) \\
&\qquad - c_{ji}(n)]\, a_{jk}(n)\, y_k(n-1).
\end{aligned} \tag{18}$$

Now by virtue of (9) we have

$$\sum_{k=1}^{r}\sum_{j=1}^{r} c_{ji}(n)\, a_{jk}(n)\, y_k(n-1) = \sum_{k=1}^{r} \rho_i(n)\, c_{ki}(n)\, y_k(n-1) = \rho_i(n)\, v_i(n).$$

As, furthermore, $\rho_i(n)$ has for $n = \infty$ the limit R_i and $c_{ji}(n+1) - c_{ji}(n)$ has the limit 0, equations (18) may be written,

$$v_i(n+1) = R_i\, v_i(n) + \sum_{k=1}^{r} e'_{ki}\, y_k(n-1),$$

where $\lim e'_{ki}(n) = 0$ for $n = \infty$. Finally, if we substitute for the $y_k(n-1)$ their values in terms of the $v_j(n)$ as indicated in (13) and remember that the d_{ji} have limits for $n = \infty$, we obtain, at least for $n > N$, the system of equations,

$$v_i(n+1) = R_i\, v_i(n) + \sum_{j=1}^{r} e_{ij}(n)\, v_j(n) \quad (i = 1, \cdots, r), \tag{B}$$

where

$$\lim_{n=\infty} e_{ij}(n) = 0.$$

This system of equations forms the basis of the subsequent work. The values of the $y_i(n)$ and hence of the $v_i(n+1)$ depend upon the initial values given to the $y_i(0)$ and also upon the coefficients $a_{ij}(p)$, $(p = 1, \cdots, n)$. By exception the $y_i(n)$ and $v_i(n+1)$ may all vanish for some values of n,

and consequently for all succeeding values. *This can happen only if the determinant of the coefficients* $a_{ij}(n)$ *vanishes for at least one value of* n. We shall assume in §2 that at least one of the $v_i(n)$ does not vanish for an infinity of values of n.

§2. *Extension of Poincaré's theorem.*

It will be supposed henceforth that

$$|R_1| > R_2| > \cdots > |R_r|. \tag{19}$$

When n increases indefinitely, some one of the r moduli $|v_i(n)|$ will be as large or larger than the remaining $r-1$ for an infinite number of values of n. I will show that this $|v_i(n)|$ will become and remain dominant over all the other $r-1$ moduli.

Suppose first that for an infinite set of values of n we have

$$|v_1(n)| \geqq |v_i(n)| \qquad (i=2, 3, \cdots, r), \tag{20}$$

and denote this set by $\{n'\}$. The division of the first of equations (B) by $v_1(n)$ and of the succeeding equations by $v_1(n+1)$ gives

$$\frac{v_1(n+1)}{v_1(n)} = R_1 + e_1(n) \tag{21}$$

and

$$\frac{v_i(n+1)}{v_1(n+1)} = \frac{R_i}{R_1+e_1(n)} \cdot \frac{v_i(n)}{v_1(n)} + \frac{e_i(n)}{R_1+e_1(n)} \qquad (i=2, \cdots, r), \tag{22}$$

in which the $e(n)$ satisfy for $n=n'$ the inequalities

$$|e_1(n')| \leqq r\bar{e}(n'), \qquad |e_i(n')| \leqq r\bar{e}(n'), \tag{23}$$

where $\bar{e}(n')$ denotes the largest of the $|e_{ij}(n')|$ in (B). Since $\lim \bar{e}(n')=0$ for $n'=\infty$, it follows that for a sufficiently large value of n' we have

$$\frac{|v_i(n'+1)|}{|v_1(n'+1)|} \leqq \frac{|R_i|}{|R_1|} + a,$$

where a is an arbitrarily prescribed positive quantity subject to the condition $|R_i|/|R_1| + a < 1$. Consequently $n'+1$ will be included with n' in the set $\{n'\}$, provided only n' is taken sufficiently large. It follows that equations (20) to (23) will hold without exception from and after some sufficiently large value of n.

From (21) we now obtain

$$\lim_{n=\infty} \frac{v_1(n+1)}{v_1(n)} = R_1. \tag{24}$$

since $\lim \bar{e}(n) = 0$. It may next be shown that

$$\lim_{n=\infty} \frac{v_i(n)}{v_1(n)} = 0. \qquad (i = 2, 3, \cdots, r). \tag{25}$$

For if this is not true, there will be at least some one value of i and some fixed magnitude $M > 0$ such that for an infinity of values of n one has

$$\frac{|v_i(n)|}{|v_1(n)|} \geqq M.$$

Call this set of values $\{n''\}$. Then since $\lim e_1(n) = \lim e_i(n) = 0$, equation (22) for $n = n''$ may be written

$$\frac{v_i(n''+1)}{v_1(n''+1)} = \left(\frac{R_i}{R_1} + e_i'(n'')\right)\frac{v_i(n'')}{v_1(n'')},$$

where $\lim e_i'(n'') = 0$ for $n'' = \infty$. Since $|R_i| < |R_1|$, this shows that when n'' is sufficiently large, the ratio $|v_i(n''+j)|/|v_1(n''+j)|$ for $j = 1, 2, \cdots$ will decrease in approximately the ratio $|R_i| : |R_1|$ until a value of j is reached for which the ratio falls below M. Thus if $|v_i(n)|/|v_i(n)|$ is equal to or greater than M for an infinity of values of n, there must also be an infinity of such values for which it falls below M. But when it falls below M, equation (22) gives

$$\frac{|v_i(n+1)|}{|v_1(n+1)|} < \frac{|R_i|}{|R_1 + e_1(n)|} M + \frac{|e_i(n)|}{|R_1 + e_1(n)|}.$$

Since now $\lim e_1(n) = \lim e_i(n) = 0$ for $n = \infty$, n may be chosen so large in the last inequality that

$$\frac{|v_i(n+1)|}{|v_1(n+1)|} < M.$$

Thus if $|v_i(n)|/|v_1(n)| < M$ for a sufficiently large value of n, it must remain so for all larger values of n. But this contradicts the hypothesis that it exceeds M for an infinity of values of n. Consequently (25) must hold, as was to be proved.

We shall next establish like results in case the $r - 1$ inequalities

$$|v_2(n)| \geqq |v_i(n)| \qquad (i \neq 2). \tag{26}$$

hold simultaneously for an infinity of values of n. Denote this particular set of values by $\{n'\}$. Then on dividing equations (B) for $i = 2$ by $v_2(n)$ and for $i \neq 2$ by $v_2(n+1)$ we obtain the following equations:

$$\frac{v_2(n+1)}{v_2(n)} = R_2 + e_2(n), \tag{27}$$

$$\frac{v_1(n+1)}{v_2(n+1)} = \frac{R_1}{R_2 + e_2(n)} \cdot \frac{v_1(n)}{v_2(n)} + \frac{e_1(n)}{R_2 + e_2(n)}, \tag{28}$$

$$\frac{v_i(n+1)}{v_2(n+1)} = \frac{R_i}{R_2 + e_2(n)} \cdot \frac{v_i(n)}{v_2(n)} + \frac{e_i(n)}{R_2 + e_2(n)} \qquad (i = 3, 4, \cdots, r), \tag{29}$$

in which the $e(n)$ for $n = n'$ satisfy the inequalities

$$|e_i(n')| \leqq r\bar{e}(n') \qquad (i = 1, 2, \cdots, r), \tag{30}$$

where $\bar{e}(n')$ denotes the largest of the $|e_{ij}(n')|$ in (B).

Equations (29) show that for a sufficiently large value of n' we have

$$|v_i(n'+1)| < |v_2(n'+1)| \qquad (i = 3, 4, \cdots, r) \tag{31}$$

since

$$|R_i| < |R_2|, \qquad |v_i(n')| \leqq |v_2(n')|, \qquad \lim_{n'=\infty} \bar{e}(n') = 0.$$

Furthermore, we must also have for a sufficiently large value of n'

$$|v_1(n'+1)| < |v_2(n'+1)|, \tag{32}$$

for otherwise the inequality

$$|v_1(n'+1)| \geqq |v_i(n'+1)| \qquad (i = 1, 3, \cdots, r)$$

would hold for an infinity of values of n'. But the latter is impossible, since it has been shown that $|v_1(n)|$ would then ultimately become and remain dominant over all the other $|v_i(n)|$, which contradicts the hypothesis that (26) is to hold for an infinity of values of n. Now (31) and (32) together show that $n' + 1$ will be included simultaneously with n' in the set $\{n'\}$, provided only n' is sufficiently large. It follows that (26)–(30) will all hold without exception from and after some fixed but large enough value of n.

From (27) we see next that

$$\lim_{n=\infty} \frac{v_2(n+1)}{v_2(n)} = R_2. \tag{33}$$

From (29) it follows also that

$$\lim_{n=\infty} \frac{v_i(n+1)}{v_2(n+1)} = 0 \qquad (i = 3, \cdots, r), \tag{34}$$

the reasoning by which this is established being exactly parallel to that by which (25) was established on our former hypothesis. Lastly, it remains to be proved that

$$\lim_{n=\infty} \frac{v_1(n)}{v_2(n)} = 0. \tag{35}$$

Suppose, if possible, this not to be true. Then there must be some magnitude

$M > 0$ which $|v_1(n)|/|v_2(n)|$ will exceed for an infinity of values of n. Denote this set of values by $\{n''\}$. For these values (28) may be written

$$\frac{v_1(n''+1)}{v_2(n''+1)} = \left(\frac{R_1}{R_2} + e'(n'')\right)\frac{v_1(n'')}{v_2(n'')},$$

where $\lim e'(n'') = 0$ for $n'' = \infty$. Since $|R_1| > |R_2|$, this shows that ultimately $n''+1$ will be included with n'' in the set $\{n''\}$. Then as i increases through the natural number system, the ratio $|v_1(n''+i)|/|v_2(n''+i|)$ will increase successively by factors approximating closer and closer to $|R_1|/|R_2|$, and hence finally we will have $|v_1(n''+i)| > |v_2(n''+i)|$ from and after some fixed value of i. But this contradicts our hypothesis (26). Hence (35) must hold, as stated.

In similar manner we may treat successively for $j = 3, 4, \cdots, r$ the hypothesis that the $r-1$ inequalities

$$|v_j(n)| \geqq |v_i(n)| \qquad (i \neq j)$$

hold for an infinity of values of n. On dividing (B) by $v_j(n)$ and $v_j(n+1)$ we obtain three sets of inequalities similar to (27), (28), (29), which furnish like results. We reach thus the following conclusion:

THEOREM 1. *If in the system of differential equations (B) we have*

$$\lim_{n=\infty} e_{ij}(n) = 0 \qquad (i, j = 1, 2, \cdots, r)$$

and we suppose

$$|R_1| > |R_2| > \cdots > |R_r|,$$

then as n increases indefinitely, either by exception the r quantities $v_i(n)$ will all vanish from and after some fixed value of n, or some one of them will ultimately dominate over the other $r-1$. If this be $v_k(n)$, then

$$\lim_{n=\infty} \frac{v_{k+1}(n)}{v_k(n)} = R_k$$

and

$$\lim_{n=\infty} \frac{v_i(n)}{v_k(n)} = 0 \qquad (i \neq k).$$

We return now to our initial equations (A). We have seen that in the relations

$$y_i(n-1) = \sum_{j=1}^{r} d_{ji}(n)\, v_j(n) \qquad (i = 1, \cdots, r) \tag{13}$$

the coefficients tend toward limiting values D_{ji}. The determinant $|D_{ji}|$ can not vanish since it is the reciprocal of $|C_{ji}| \neq 0$. Consequently for each value of j not all of the D_{ji} $(i = 1, \cdots, r)$ can vanish. Consider now

the equation

$$\frac{y_i(n)}{y_i(n-1)} = \frac{\sum_{j=1}^{r} d_{ji}(n+1)\, v_j(n+1)}{\sum_{j=1}^{r} d_{ji}(n)\, v_j(n)}. \tag{36}$$

If, as in Theorem 1, $v_k(n)$ denotes that one of the $v_i(n)$ which ultimately becomes dominant, and if D_{kl} denotes one of the D_{kj} $(j = 1, \cdots, r)$ which does not vanish, it follows by application of the theorem to (36) that

$$\lim_{n=\infty} \frac{y_l(n-1)}{y_l(n)} = \lim \frac{d_{kl}(n+1)\, v_k(n+1)}{d_{kl}(n)\, v_k(n)} = \lim \frac{v_k(n+1)}{v_k(n)} = R_k.$$

Also we have

$$\lim_{n=\infty} \frac{y_i(n)}{y_l(n)} = \lim \frac{d_{ki}(n+1)\, v_k(n+1)}{d_{kl}(n+1)\, v_k(n+1)} = \frac{D_{ki}}{D_{kl}}. \qquad (i \neq l).$$

This shows that the point $y_i(n)$ $(i = 1, \cdots, r)$ for $n = \infty$ approaches as its limit one of the invariant points of (15). This result may be restated as follows:

THEOREM 2. *Let (A) be a system of difference equations in which the coefficients $a_{ij}(n)$ have limiting values A_{ij} for $n = \infty$. If the roots R_i of the limiting characteristic equation (7) are no two of equal modulus, then unless by exception the $y_i(n)$ all vanish from and after some fixed value of n, there is at least one function $y_l(n)$ in the solution $y_j(n)$, $(j = 1, \cdots, r)$, for which we have*

$$\lim_{n=\infty} \frac{y_l(n)}{y_l(n-1)} = R_k,$$

where R_k is one of the roots of (7).

If also (A) is a sequence of non-singular collineations in r homogeneous variables, which have for $n = \infty$ a limiting form, then any initial point $y_1(0)$, $\cdots y_r(0)$ by the successive application of the collineations will approach as a limit one of the invariant points of the limiting collineation (15).

The theorem of POINCARÉ is an immediate consequence of the first part of this theorem, for by the very construction of (3) we have

$$\lim_{n=\infty} \frac{u_i(n)}{u_i(n+1)} = \lim \frac{u_{i-1}(n)}{u_{i-1}(n-1)} \qquad (i = 2, \cdots, r).$$

To guard against misconception I will add in conclusion that not all of the r ratios $y_i(n)/y_i(n-1)$ need have limits for $n = \infty$, and even if they have, not all of the limits need to be roots of (7). The truth of the latter

statement will be obvious from a consideration of the special system,

$$v_1(n) = R_1 v_1(n-1),$$
$$v_2(n) = e(n) v_1(n-1) + R_2 v_2(n-1)$$

in which $e(n)$ is an infinitesimal having the value

$$(S_1 - R_2)\frac{\bar{v}_2(n-1)}{\bar{v}_1(n-1)} \qquad (0 < R_2 < S_1 < R_1).$$

Here the limit of $v_2(n)/v_2(n-1)$ for the particular integral $v_i(n) = \bar{v}_i(n)$ is an arbitrarily chosen constant S_1.

UNIVERSITY OF WISCONSIN.

ONE-PARAMETER PROJECTIVE GROUPS AND THE CLASSIFICATION OF COLLINEATIONS*

BY

EDWARD B. VAN VLECK

The origin of the following paper is to be found in the application of the iterative principle to the classification of collineations. To this I have been unexpectedly led by my preceding paper on the *Extension of a Theorem of Poincaré*. Other topics closely connected with my subject which will be treated here are as follows:

1. The determination of a normal form for one-parameter collineation groups in n homogeneous variables, also for homogeneous linear groups in n variables.

The form consists namely of components of the structure (3) below, where m is the parameter of the group. The total number of equations of all the components must, of course, be equal to n.

2. The determination of a *real* normal form for *real* groups of the character above described.

It consists of components of form (3), in which ρ is to be taken positive, and of components of the form (14).

3. The enumeration and classification of these groups for any value of n; in particular, for $n = 3, 4$.

4. The varieties and form of the real path curves (the so-called W-curves) for real one-parameter collineation groups in space of three dimensions. The quadratic complex of tangents is also briefly considered.

The prevalent mode of classifying collineations has been developed by SEGRE and others and is based upon the normal form into which the *finite* equations of collineation may be transformed in accordance with the theory of elementary divisors. Geometrically, this form is dependent upon the invariantive configuration of points, lines, planes, etc. LIE, on the other hand, uses the *infinitesimal* transformation for classification.† The results correspond to those obtained by the preceding method. An obvious inconvenience or difficulty arises when a collineation is given, as is most commonly

* Presented under a slightly different title at the Chicago meeting of the Society, April 29, 1911.

† Cf., for example, LIE-SCHEFFERS, *Vorlesungen über continuierliche Gruppen*, pp. 65, 290–291.

the case, in finite and not in infinitesimal form. So far as I am aware, there has not existed prior to the present paper any criterion or formula giving the infinitesimal transformation, or transformations by which a given finite collineation may be generated — for example, a collineation with given numerical coefficients. For the case of plane collineations a passage from the given finite form to the infinitesimal transformation was effected as long ago as 1871 by KLEIN and LIE,* but their method has not been developed for space of higher dimensions.

LIE's work also opens up the possibility of another mode of classification based upon the character of the system of path curves to which an infinitesimal transformation gives rise. This method has been largely ignored; indeed, I do not feel sure that anyone has had expressly in mind the use of path curves for the purpose of *classifying* collineations. Though less direct, this method would have the advantage of being more discriminating than either of the two preceding methods. For example, while the method of SEGRE gives five types of plane collineations, this method at once subdivides one of his five types into three,† thus yielding seven distinct types in all. A complete classification of the path-curve systems in space S_3, and correspondingly of space collineations, has not been given hitherto, though most interesting properties of some of these curve systems have been given by KLEIN and LIE.‡

The method of classification which I shall give here has the same discriminating power as the standpoint last mentioned, yet retains the advantages of the method of SEGRE. My starting point is, in fact, his familiar classification. The various types of collineations are iterated indefinitely, when differences arise between collineations of the same type, thus giving a further subdivision into classes. The iteration supplemented by functional interpolation also furnishes a one-parameter collineation group and leads to a very simple solution of the following problem: *Given a finite collineation in n homogeneous variables, to set up in finite form the continuous one-parameter group (or groups) in which it is contained.* Thereby the classification of collineations becomes parallel to the classification of these groups or of the accompanying system of path curves, except for the very restricted set of cases in which a collineation can be included in several groups.

*Mathematische Annalen, vol. 4, p. 50. See also BÔCHER, Bulletin of the New York Mathematical Society, vol. 1 (1892), p. 225.

†Namely, the type with three invariant points. In LIE-SCHEFFERS (loc. cit., pp. 77–79) a distinction of cases is made according as the three invariant points are real or one real and the other two conjugate imaginaries. In the latter case the path curves are in general logarithmic spirals. No note is there made of the fact that a seventh case arises when the system of spirals degenerates into a system of concentric circles, although the existence of this case is obvious from the equations given. This failure to specifically point out the existence of a seventh case is indicative of the general neglect of the application of Lie's path curves to the classification of collineations.

‡Comptes Rendus, vol. 70 (1870), pp. 1222, 1275.

Since reaching my result I have found that precisely the same iterative method was previously applied in a very well known memoir by KLEIN and LIE* to pass from the five types of plane collineations to the corresponding continuous one-parameter groups. Indeed, it seems to have been precisely this which led to the introduction of the infinitesimal transformation.† It seems therefore rather strange that the iterative principle has not been similarly applied to the general case of n variables. Possibly the discovery and development of the infinitesimal transformation with its wealth of applications diverted work from this direction. Possibly also the generalization may have been overlooked because the plane collineations iterated by KLEIN and LIE were expressed in unhomogeneous variables and not in the standard homogeneous form in which they are given today by the theory of elementary divisors.

It is to be observed that the program here brought forward for one-parameter collineation groups is the reverse of that of LIE. The method of LIE, on the one hand, starts with the infinitesimal transformation and ascends by integration, etc., to continuous groups of finite collineations. Here, on the other hand, a single finite collineation is taken as the starting point, from which the continuous group and infinitesimal transformation are to be derived. The normal form obtained for the group conducts at once to the complete enumeration of collineation groups in finite form for any number of homogeneous variables, a problem which seems to have been solved hitherto only for the cases of two and of three variables.

My formulas for the continuous groups give also the parametric representation of the corresponding systems of W-curves. *A brief study is made of the different classes of real W-curves in S_3*, some of which are new and very interesting.

For the particular case of the collineation of a straight line into itself, given by the formula

$$x' = \frac{ax + b}{cx + d},$$

we have in the literature the familiar division into parabolic, elliptic, hyperbolic and loxodromic types. These distinctions arise from the interpretation of the imaginary path curves upon the line as real curves in the complex plane of x. The classification of collineations which I have introduced here may be regarded as essentially a corresponding classification for any number of variables, though no effort has been made to interpret unreal path curves in n-dimensions as real path curves in $2n$-dimensional space. Such an inter-

*Mathematische Annalen, vol. 4 (1871), p. 50.

†Cf. Mathematische Annalen, vol. 53 (1900), pp. 6–8. In his review of LIE's work NOETHER expresses his regret that the program of KLEIN and LIE for three-dimensional collineations sketched in the Comptes Rendus, was not later completed, as was projected.

pretation would lead to a subdivision of some of my classes in correspondence with the cleavage between the hyperbolic and loxodromic types in the complex plane. The two latter conduct by iteration any point of the complex x-plane to one of the two invariant points as a limit, but are commonly differentiated from one another because of the differences in the system of path curves to which they give rise.

It may be useful in this connection to refer the reader to some recent work by J. WESLEY YOUNG.* By the use of "chains" YOUNG extends the notion of path curves in the complex plane of x in a very different manner to the case of two complex variables.

PART I. THE ONE-PARAMETER COLLINEATION GROUP.

§ 1. *The standard form for a collineation.*—Consider any linear substitution with non-vanishing determinant given by the formula:

$$(C') \qquad \begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n, \\ &\cdots\cdots\cdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n. \end{aligned}$$

This will be interpreted at pleasure either as the general representation of a collineation expressed in n homogeneous variables or as a homogeneous collineation in space of n dimensions.

Suppose now that C' is transformed by application of a second linear homogeneous transformation T in n variables with non-vanishing determinant. The theory of elementary divisors shows that by properly choosing T the transformed substitution $C = TC'T^{-1}$ can be brought into simplified standard form. In this form the variables of C are separated into independent sets, only the variables within a set being connected with one another. If the variables of a set are denoted by $x_1, x_2, \cdots, x_r$, the equations connecting them have the character:

$$(1) \qquad \begin{aligned} x_1' &= \rho x_1, \\ x_2' &= x_1 + \rho x_2, \\ x_3' &= x_2 + \rho x_3, \\ &\cdots\cdots \\ x_r' &= x_{r-1} + \rho x_r \end{aligned} \qquad (r \leqq n).$$

Thus the substitution (collineation) is resolved into *component substitutions* (*collineations*) which may be considered in entire independence of one another. I shall call r the *order* of the component. The sum of the orders is equal to n.

* These Transactions, vol. 11 (1910), p. 280. On the basis of his chains YOUNG distinguishes six classes of plane collineations, one of which could have been readily split into two.

The values of ρ in the component substitutions are the roots of the characteristic equation

$$(2) \qquad \begin{vmatrix} a_{11}-\rho & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\rho & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\rho \end{vmatrix} = 0.$$

To each elementary divisor $(x-\rho_i)^r$ of this determinant there will correspond a component of order r. If for a multiple root ρ_i there are several elementary divisors, there will be an equal number of component substitutions having the same value of ρ_i. Inasmuch as the elementary divisors are not altered when C' is transformed through T, the standard form obtained for C is unique; but there is sometimes the possibility of reaching it by application of essentially distinct transformations T, or, in other words, by different changes of variable T.

§ 2. *Iteration of the collineation.*—Suppose now the original collineation C' to be repeated m times. Then

$$C'^m = T^{-1} C^m T.$$

Thus the problem of iterating C' is equivalent to that of iterating C, the result in either case being obtained from that in the other by transformation through T or its inverse.

Consider then the effect of repeating m times a component collineation (1). The result is given by the equations:

$$(3) \qquad \begin{aligned} x_1' &= \rho^m x_1, \\ x_2' &= m\rho^{m-1} x_1 + \rho^m x_2, \\ x_3' &= \frac{m(m-1)}{2!}\rho^{m-2} x_1 + m\rho^{m-1} x_2 + \rho^m x_3, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_r' &= \frac{m(m-1)\cdots(m-r+2)}{(r-1)!}\rho^{m-r+1} x_1 \\ &\qquad + \frac{m(m-1)\cdots(m-r+3)}{(r-2)!}\rho^{m-r+2} x_2 + \cdots + \rho^m x_r. \end{aligned}$$

This can be established immediately by mathematical induction. For if we repeat the substitution once more, putting

$$x_1'' = \rho x_1', \qquad x_2'' = x_1' + \rho x_2', \qquad \cdots, \qquad x_r'' = x_{r-1}' + \rho x_r',$$

we obtain equations for the x_i'' in terms of the x_i which are the same as (3) but with $m+1$ in place of m.

One consequence of the form of (3) is so immediate as to be worthy of note in passing, although not a new result. Suppose C' and therefore C to be a periodic substitution in n variables so that $C^p \equiv 1$. The first equation of (3) shows that ρ must be a pth root of unity. In the second equation it is impossible for x_2' and x_2 to be identical for $m = p > 1$. Consequently a necessary condition for a periodic substitution is that all the roots ρ_i of the characteristic equation shall be pth roots of unity and that every elementary divisor shall be of first degree. This condition is also obviously sufficient.

§ 3. *Introduction of a continuous parameter.*—When m is not a positive integer, the component substitutions (3), taken together, still define a substitution which I shall denote by C^m. To remove the ambiguity lurking in ρ^m, it shall be understood that ρ is expressed in the form $|\rho| e^{i\vartheta}$ when $2n\pi \leqq \vartheta \leqq 2(n+1)\pi$ (n being an integer), and the mth power is formed by the usual rule.

Consider next the effect of performing a substitution C^h after C^k. The resulting substitution is seen at once to be made up of components of the form:

$$
\begin{aligned}
x_1'' &= \rho^{h+k} x_1, \\
x_2'' &= c_{21} \rho^{h+k-1} x_1 + \rho^{h+k} x_2, \\
&\cdots\cdots\cdots\cdots \\
x_r'' &= c_{r1} \rho^{h+k-r+1} x_1 + c_{r2} \rho^{h+k-r+2} x_2 + \cdots + c_{r,r-1} \rho^{h+k-1} x_{r-1} + \rho^{h+k} x_r,
\end{aligned} \tag{4}
$$

in which the coefficients c_{ij} are polynomials in h and k. Put now $m = h + k$ in (3) and compare the coefficients of (3) and of (4). We know that any two corresponding coefficients are equal to one another for positive integral values of h and k. If h only is given a positive integral value, the two coefficients are identical polynomials in k, inasmuch as they are equal to one another for an infinity of values of k. Consequently corresponding coefficients in these two polynomials in k are equal. But being equal for all positive integral values of h, they must be identical polynomials in h. We conclude therefore that for $m = h + k$ corresponding coefficients in (3) and (4) are identical. Thus for all values of h and k we have

$$C^{h+k} = C^h C^k = C^k C^h. \tag{5}$$

In particular, C^{-h} is the inverse of C^h since $C^0 \equiv 1$.

§ 4. *The p-th root of a collineation.*—Let p denote henceforth a positive integer. To obtain a linear substitution whose pth power is a given substitution C' we have merely to put $m = 1/p$ and take $T^{-1} C^{1/p} T$. In each component (3) of $C^{1/p}$ we may choose for $\rho_i^{1/p}$ any one of the pth roots of ρ_i. Hence if C is made up of q component substitutions, we obtain in this manner p^q substitutions whose pth power is the given substitution.

The question now arises whether all possible roots of a given substitution or collineation are thus obtained. That this is not always the case is apparent on taking the identical collineation $x_i' = \rho x_i \, (i = 1, \cdots, n)$ as the given one. Indeed, we shall now see that whenever C contains two component substitutions *having equal values of* ρ, there must be an infinity of distinct pth roots. Suppose $x_1, \cdots, x_r$ and $y_1, \cdots, y_s \, (s \geqq r)$ to be the variables belonging to two such components. Then without destroying the normal form (1) the r variables x_i can be replaced by $z_i \equiv x_i + k y_i \, (i = 1, \cdots, r)$, where k denotes an arbitrary constant. Let C be denoted by C_z after these z_i-coördinates have been introduced. Clearly $C_z^{1/p}$ is a pth root* of C which contains the substitutions

$$z_1' = \rho^{1/p_1} z_1, \qquad y_1' = \rho^{1/p_2} y_1,$$

where ρ^{1/p_1}, ρ^{1/p_2} denote pth roots of ρ. If different pth roots are here selected, the only planes of the pencil $z_1 + c y_1 = 0$ which are invariant under the transformation $C^{1/p}$ are $z_1 = 0$ and $y_1 = 0$. Now the position of $z_1 = 0$ varies with the choice of k. Hence by varying its value we obtain an infinity of different pth roots $C^{1/p}$. It will be shown later that only in the class of cases under consideration will the number of pth roots of C, and therefore also of C', exceed p^q. By way of anticipation we may therefore state the following conclusion:

If q is the number of elementary divisors of (2), *the number of p-th roots of a given substitution C' is equal to p^q, unless there are two elementary divisors having equal roots ρ, when the number of p-th roots will be infinite.*†

When C' is interpreted as a collineation in space of $n - 1$ demensions, the number p^q is to be replaced by p^{q-1} inasmuch as p substitutions correspond to one collineation owing to the homogeneous character of the variables.

This result is, however, only incidental. Our concern here is not with the number of pth roots but rather with the form into which C and any one of its pth roots can be simultaneously thrown. I shall show that by properly choosing the coördinate planes of reference any particular root under consideration may be put into the form $C^{1/p}$ without destroying the normal form of C. For brevity of statement it will be said that *any substitution* (*collineation*) *has the normal form* (1) *if each component substitution* (*collineation*) *is of this form.*

Denote by K any particular pth root of C under consideration, and suppose first that K itself has first been thrown into normal form (1). Then C as its pth iterative has the form (3) where $m = p$. Make now simultaneously in

* $C^{1/p}$ is used throughout to denote, not any collineation whose pth power is C, but one which is obtained by putting $m = 1/p$ in each component (3) of C^m.

† The pth roots of a given matrix or substitution have been considered in a very different manner by SYLVESTER; Comptes Rendus, vol. 94 (1882), pp. 55, 396. SYLVESTER'S results for the number of pth roots is not complete. He obtains the number only for the simple case $q = n$, and for a certain class of cases for which the number is infinite.

each component of K and C a change of variable of the form:

$$
\begin{aligned}
x_1 &= y_1, \\
x_2 &= y_1 + c_{22}\, y_2, \\
x_3 &= y_1 + c_{32}\, y_2 + c_{33}\, y_3, \\
&\cdots\cdots\cdots\cdots \\
x_r &= y_1 + c_{r2}\, y_2 + \cdots + c_{rr}\, y_r,
\end{aligned}
\qquad (c_{ii} \neq 0). \tag{6}
$$

and determine the c_{ij} so that the component of C shall be transformed into normal form:

$$
\begin{aligned}
y_1' &= \rho^p\, y', \\
y_i' &= y_{i-1} + \rho^p\, y_i \qquad (i = 2, 3, \cdots, r).
\end{aligned} \tag{7}
$$

We shall see first that the c_{ij} are thereby uniquely determined. For this purpose let the equations (3) for C (wherein $m = p$) be abridged by

$$x_i' = d_{i1}\, x_1 + d_{i2}\, x_2 + \cdots + d_{i,\,i-1}\, x_{i-1} + \rho^p\, x_i \qquad (i = 1, \cdots, r).$$

Then by the change of variables (6) they become

$$
\begin{aligned}
&c_{ii}\, y_i' + c_{i,\,i-1}\, y_{i-1}' + \cdots + c_{i2}\, y_2' + y_1' \\
&\qquad = \sum_{j=1}^{i-1} d_{ij}\, (y_1 + c_{j2}\, y_2 + \cdots + c_{jj}\, y_j) + \rho^p\, (y_1 + \cdots + c_{ii}\, y_i).
\end{aligned}
$$

If, now, all the c_{jk} $(j \leqq i-1)$ have been determined so that the first $i-1$ of equations (7) are fulfilled, the equations just written become in consequence

$$c_{ii}\, y_i' = \sum_{j=1}^{i-1} d_{ij}\, (y_1 + c_{j2}\, y_2 + \cdots + c_{jj}\, y_j) + \rho^p\, c_{ii}\, y_i - \sum_{j=2}^{i-1} c_{ij}\, y_{j-1}.$$

To make this equation identical with the ith relation of (7) we must put:

$$
\begin{aligned}
c_{i2} &= \sum_{j=1}^{i-1} d_{ij}, \\
c_{ik} &= \sum_{j=k-1}^{i-1} d_{ij}\, c_{j,\,k-1} \qquad (k = 3, \cdots, i-1), \\
c_{ii} &= d_{i,\,i-1}\, c_{i-1,\,i-1} \neq 0 \qquad (c_{11} = 1).
\end{aligned}
$$

These equations therefore determine uniquely $c_{i2}, c_{i3}, \cdots, c_{ii} \neq 0$, after the $c_{i-1,\,j}\, c_{i-2,\,j}, \cdots$ have been already similarly determined. Thus the coefficients of (6) are uniquely determined by the requirement the C shall be transformed into normal form (7), as was to be proved.

It remains to ascertain what K will simultaneously become in consequence of the transformation. Since by hypothesis K was initially in normal form

(1), the change of variables (6) throws its components into the form:

$$(8)\qquad \begin{aligned} y_1' &= \rho y_1, \\ y_i' &= b_{i1}\, y_1 + \cdots + b_{i,\,i-1}\, y_{i-1} + \rho y_i \qquad (i = 2, \cdots, r), \end{aligned}$$

where the coefficients b_{ij} are yet to be ascertained. Obviously the pth iterative of (8) has the same form as (8) itself, ρ being replaced by ρ^p and the other coefficients by polynomials in terms of the b_{ij}. We know further that the pth iterative C has the form (7). If now in these two expressions for this iterative we equate corresponding coefficients, it will be found that we obtain a set of equations of the following character:

$$\begin{aligned} pb_{21}\,\rho^{p-1} &= 1; \\ pb_{32}\,\rho^{p-1} &= 1, \\ pb_{31}\,\rho^{p-1} + [\quad\quad] &= 0; \\ pb_{43}\,\rho^{p-1} + [\quad\quad] &= 1, \\ pb_{42}\,\rho^{p-1} + [\quad\quad] &= 0, \\ pb_{41}\,\rho^{p-1} + [\quad\quad] &= 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot &, \end{aligned}$$

where in each line the bracket stands for a polynomial in the b_{ij} of the previous lines. Hence the b_{ij} are uniquely determined, and consequently there is but a *single* substitution (8), with given ρ, whose pth power is (7). Now (7) is the same as (1) with ρ^p in place of ρ. Consequently if we replace ρ by ρ^p in (3), put $m = 1/p$, and then substitute y_i, y_i' for x_i, x_i', we shall obtain a pth root of form (8) with the given ρ. As this root was obtained from (3) by placing $m = 1/p$, the statement has been thus finally verified that *C can be so transformed into normal form* (1) *that any particular one of its p-th roots shall be simultaneously of form $C^{1/p}$.*

§ 5. *The derivation of the one parameter collineation group.*—From equation (5) it is evident that we obtain a continuous one-parameter group C^m by varying m continuously through real values in (3). For brevity it will be said that any group has the normal form (3) when it consists of components of this form. The question next to be considered is whether every continuous one-parameter collineation group can be thrown into this normal form.

In the consideration of this question it will be assumed that if any substitution is contained in the group, its inverse and one of its square roots are also contained therein. Thus with any particular substitution C, assumed in normal form, will be included a succession of 2^N-th roots ($N = 1, 2, \cdots$). Our aim is to establish next that this entire succession of roots may be simul-

taneously thrown into the form C^m ($m = 1/2^N$) by proper choice of the coördinate planes.

To this end consider first what change of variables

$$y_i = k_{i1}x_1 + \cdots + k_{in}x_n \qquad (i = 1, \cdots, n) \tag{9}$$

can be made without destroying the normal form of C. In terms of the y_i the substitution C is to consist of exactly the same number of components with the same orders as in terms of the x_i. Take a sample component

$$y_1' = \rho' y_1, \qquad y_i' = y_{i-1} + \rho' y_i \qquad (i = 2, \cdots, s), \tag{10}$$

in which to make the substitution (9). Since the x-variables were separated in C into independent sets (1), our equations (10) after the change of variables (9) may be split up into sets of equations, each involving only the x-variables of a single component collineation. Thus corresponding to (1) we obtain the s equations:

$$\begin{aligned} k_{11}x_1' + \cdots + k_{1r}x_r' &= \rho'(k_{11}x_1 + \cdots + k_{1r}x_r), \\ k_{i1}x_1' + \cdots + k_{ir}x_r' &= \rho'(k_{i1}x_1 + \cdots + k_{ir}x_r) \\ &\quad + (k_{i-1,1}x_1 + \cdots + k_{i-1,r}x_r) \qquad (i = 2, \cdots, s). \end{aligned}$$

If the values for x' given in (1) are now substituted, these equations become identities. Unless $\rho' = \rho$, we see that in the first of these identities every $k_{1i} = 0$, and then from the succeeding ones that every k_{ij} vanishes. In other words, $y_1, \cdots, y_s$ can involve only the x-variables of component substitutions which have the same value of ρ. When $\rho = \rho'$, the identities give

$$0 = k_{12} = k_{13} = \cdots = k_{1r}, \qquad k_{i-1,r} = 0, \qquad k_{ij} = k_{i-1,j-1} \qquad (i = 2, \cdots, s;\ j = 2, \cdots, r).$$

Hence the part of $y_1, \cdots, y_s$ which involves $x_1, \cdots, x_r$ is as follows:

$$\begin{aligned} \bar{y}_1 &= k_{ss}x_1, \\ \bar{y}_2 &= k_{s,s-1}x_1 + k_{ss}x_2, \\ &\cdots\cdots\cdots\cdots \qquad (k_{si} = 0 \text{ if } s > r). \\ \bar{y}_{s-1} &= k_{s2}x_1 + k_{s3}x_2 + \cdots + k_{ss}x_{s-1}, \\ y_s &= k_{s11}x_1 + k_{s2}x_2 + \cdots + k_{s,s-1}x_{s-1} + k_{ss}x_s. \end{aligned} \tag{11}$$

The change of variables (9) therefore consists of components

$$y_i = \Sigma \bar{y}_i \qquad (i = 1, \cdots, s),$$

where the y_i summed are expressions (11), each of which contains the x_i of a single component (1) having $\rho = \rho'$.

We proceed next to consider the effect of this change of variables upon

$C^{1/p}$. Suppose that C comprises q components, corresponding to the values

$$\rho_1, \ \rho_2, \ \cdots, \ \rho_q. \tag{12}$$

Then the corresponding sequence of roots for $C^{1/p}$ is

$$\rho_1^{1/p}, \ \rho_2^{1/p}, \ \cdots, \ \rho_q^{1/p}. \tag{13}$$

In case any two members ρ_i, ρ_j of (12) are equal, we shall take the corresponding roots in (13) equal. Then it will be shown quickly that our change of variables which left C unaltered in form must also leave the form of this $C^{1/p}$ unaltered. First, it is clear that any change of variable (11) leaves the first s equations of (3) unaltered in form, for if we multiplied the first l equations of (3) successively by $k_{s,\,s-l+1}$, $k_{s,\,s-l+2}$, $\cdots$, k_{ss} and add, we obtain again the lth equation of (3) with $\bar{y}_i'$, $\bar{y}_i$ in place of x_i', x_i. Secondly, if we have several sets of variables $\bar{y}_i'$, $\bar{y}_i$ satisfying the same set of equations of form (3), then the sums of corresponding variables $\Sigma\bar{y}_i'$, $\Sigma\bar{y}_i$ also satisfy such a set of equations. It follows therefore that we have exactly the same equations for $C^{1/p}$ in terms of the y_i as in terms of the x_i. Accordingly the form of $C^{1/p}$ is unaltered.

Consider next the totality of pth roots of C ($p = 1, 2, \cdots$), which fulfill the condition that any two members of the sequence (13) are equal when the corresponding members of (12) are equal. We have found above that without destroying the normal form of C any one of these roots may be brought into the normal form $C^{1/p}$, and we have just proved that then any change of variable which does not affect the form of C will also not destroy the normal form $C^{1/p}$, provided the sequences (12) and (13) are alike in respect to the equality or inequality of their terms. It follows that the entire set of roots now under consideration will simultaneously assume the normal form $C^{1/p}$ whenever C itself has the normal form (1). In particular, if no two members of the sequence (12) are equal, *every* pth root ($p = 1, 2, \cdots$) will have the form $C^{1/p}$. Now for a fixed value of p the number of roots of this form is p^q, since there are in all q components (3) and in each we have a choice of p values for $\rho^{1/p}$. Thus we have established the conclusion italicized in § 4, to the effect that when no two members of (12) are equal, the number of pth roots of C is exactly p^q.

We are now ready to proceed with the derivation of the normal form of a continuous one-parameter collineation group. With a given collineation C, assumed in normal form, there is also included in the group a succession of 2^nth roots ($n = 1, 2, \cdots$) of C. Since each of these is a square root of the preceding, each $\rho_i^{1/2^{n+1}}$ in (13) is a square root of the corresponding $\rho_i^{1/2^n}$. Hence as n increases indefinitely, we reach a point subsequent to which any two members of (13), if still equal, henceforth remain so. Suppose the permanent status to be reached for $n \geqq m$. Then let K denote the particular 2^mth

root of C which is included in our succession of its roots. When K is thrown into normal form (1), all the succeeding roots by the last paragraph must have the form $K^{1/p}$ where $p = 2^{n-m}$. Also C as the mth iterative of K has the form K^m. Suppose now that C by a change of variable, as explained in § 4, is brought back into normal form (1). Then K takes the form $C^{1/2^m}$ and its ith powers the form $C^{i/2^m}$. Its successive pth roots ($p = 2^{n-m}$) before the change of variable (6) had the form (8), with $\rho^{1/2^n}$ in place of ρ, and they remain of the same form after the transformation. But in § 4 we saw that there was but a single substitution of such form, with preassigned values of $\rho_i^{1/2^n}$, whose 2^nth power was C, and that substitution had the form $C^{1/2^n}$. It follows that *our succession of 2^nth roots of C ($n = 1, 2, \cdots$) by proper choice of the coördinate system may all be put simultaneously into the standard form $C^{1/2^n}$.*

We now have in our group all substitutions of the form $C^{1/2^n}$ and with them their pth iteratives $C^{p/2^n}$ and their inverse substitutions $C^{-p/2^n}$. Thus our group has been found to contain substitutions of form C^m for a dense set of values of m, namely $m = \pm p/2^n$ ($p, n = 1, 2, \cdots$). It remains to extend this conclusion to all real values of m. By hypothesis the group is continuous. Hence in the first equation of (3),

$$x_1' = \rho^m x_1,$$

such a succession of roots ρ^m must be selected for $m = p/2^n$ as will yield a continuous function. Obviously a unique continuous modulus $|\rho|^m$ is thereby determined. Let $f(m)$ denote the real part of $e^{i \arg \rho^m}$. From equation (5) it follows that for values of h and k selected from the dense set $\{\pm p/2^n\}$ $f(m)$ must satisfy the cosine law, and accordingly

$$f(h+k) - f(h-k) = 2f(h)f(k).$$

Since also $f(m)$ is continuous, this equation must consequently subsist for all real values of h and k. But the real and continuous functions $f(x)$ which satisfy this equation and do not exceed 1 in absolute value were shown by Cauchy to be $f(x) = 0, 1, \cos cx$. Consequently for real values of m the real part of $e^{i \arg \rho^m}$ is $\cos(m \arg \rho)$, and hence the group contains the substitution $x_1' = \rho^m x_1$.

In every equation of (3) subsequent to the first, ρ^m appears in the coefficients multiplied by a polynomial in m with real coefficients. As the equations subsist for the dense set of values $m = \pm p/2^n$, it follows from considerations of continuity that they must also hold for all real values of m. We thus reach the following conclusion:

Every continuous one-parameter collineation group in n homogeneous variables with real parameter m, or homogeneous one-parameter linear substitution group in n variables, may be thrown by proper choice of coördinates into a normal form consisting of components of the form (3).

For $m = 0$ the equations (3) give the identical transformation. The infinitesimal transformation which generates the group may be obtained by assuming m to be infinitesimal. For convenience put $m = dt$, then on expanding the coefficients of (3) in ascending powers of dt and neglecting all but the first power of dt we obtain the increments*

$$dx_i = x_i' - x_i = dt\left[\frac{(-1)^i x_1}{(i-1)\,\rho^{i-1}} + \frac{(-1)^{i-1} x_2}{(i-2)\,\rho^{i-2}} + \cdots + \frac{x_{i-1}}{\rho} + x_i \log \rho\right] \qquad (i = 1, \cdots, r).$$

The reality of the infinitesimal transformation will be discussed later.

§6. *Lists of one-parameter collineation groups.*—Our conclusion concerning the form of continuous one-parameter collineation groups may be applied at once to the construction of a complete list of such groups for space of $n - 1$ dimensions. For this purpose we write down first in all possible ways sets of component collineations of the form (3) for which the sum of the orders r will be equal to n. Then under each set of n equations so constructed we are to distinguish all possible cases which may arise through the equality or non-equality of the ρ_i of the component collineations, inasmuch as these cases differ from one another in respect to the character of the invariantive configuration of points, lines, planes, etc. Thus for $n = 3$ we obtain in addition to the identical transformation the following types of one-parameter collineation groups (m = parameter):

Type I.
$$x_1' = \rho_1^m x_1,$$
$$x_2' = \rho_2^m x_2,$$
$$x_3' = \rho_3^m x_3 \qquad (\rho_1 \neq \rho_2 \neq \rho_3).$$

Type II.
$$x_1' = \rho_1^m x_1,$$
$$x_2' = m\rho_1^{m-1} x_1 + \rho_1^m x_2,$$
$$x_3' = \rho_3^m x_3 \qquad (\rho_1 \neq \rho_3).$$

Type III.
$$x_1' = \rho^m x_1,$$
$$x_2' = m\rho^{m-1} x_1 + \rho^m x_2,$$
$$x_3' = \frac{m(m-1)}{2!}\rho^m x_1 + m\rho^{m-1} x_2 + \rho^m x_3.$$

Type IV. Same as I but with $\rho_2 = \rho_3$.

Type V. Same as II but with $\rho_1 = \rho_3$.

* Taking all such sets of increments we obtain a canonical form for the symbol of an infinitesimal collineation more explicit than that given in LIE-ENGEL'S *Theorie der Transformationsgruppen*, vol. 1, top of p. 585.

These five types of groups correspond to the familiar five varieties of plane collineations which result on placing $m = 1$. In the literature they occur mostly in unhomogeneous form. One such form results on dividing the last two of our three equations by the first and is as follows:*

(a = parameter, k = constant)

I. $x' = ax,$
$y' = a^k y \ (k \neq 1).$

II. $x' = a + x,$
$y' = k^a y \ (k \neq 1).$

III. $x' = x + a,$
$y' = y + ax + \dfrac{a(a-k)}{2}.$

IV. $x' = ax,$
$y' = ay.$

V. $x' = x + a,$
$y' = y.$

A corresponding enumeration of the 13 types of one-parameter collineation groups in three dimensions will be found in the course of §12. While such an enumeration for S_3 has been given previously;† it is to be emphasized that the advantage of the present results is that they can be applied with equal ease and directness to a like enumeration of one-parameter collineation groups *in finite form for n-dimensional space.* So far as I know, this has not been previously attained.

§ 7. *Homogeneous one-parameter groups in n variables.*—A second application of the result of § 5 is the construction of list of homogeneous linear groups in n variables. In this case certain distinctions must be considered, which do not arise in the case of homogeneous coördinates and which are due to the interpretation of the one-parameter group as an affine group in S_n, whereby the plane at infinity is left unaltered. These distinctions are perceived on bringing in an $(n+1)$-th variable to render the coördinates homogeneous. This merely adds to our system another equation, $x'_{n+1} = \rho_{n+1} x_{n+1}$, in which $\rho_{n+1} = 1$. Hence the character of the complete invariantive configuration of the group is not merely contingent upon the system of equalities and inequalities which subsist between the ρ_i of the component collineations, but it also depends upon their equality or non-equality to the special value $\rho_{n+1} = 1$. A second point of difference is that the collineation $x'_i = \rho x_i \ (i = 1, \cdots, n)$ is no longer to be neglected, since it no longer furnishes the identical transform-

* Cf. Fr. Meyer, *Mathematical papers read at the International Mathematical Congress, Chicago,* 1893, pp. 189–190. The above forms can be obtained by integrating the five types of plane infinitesimal collineations given by Lie-Scheffers, p. 65. In type III the term $-ak/2$ can be removed by change of variable $x = \bar{x} + k/2$.

† Given in unhomogeneous form by Newson, Kansas University Quarterly, July, 1898, pp. 140–141.

ation. As a specimen application I shall enumerate the homogeneous one-parameter groups for $n = 3$.

Types I–V are the same as the types I–V enumerated in § 6 but with the added restriction that each $\rho_i \neq 1$. We have also in addition

Type VI. $$x_i' = \rho^m x_i \qquad (\rho \neq 1;\ i = 1, 2, 3).$$

The remaining types are obtained from these by placing one or more of the ρ_i equal to 1. We may neglect all cases in which one coördinate remains unaltered, provided the changes in the other two are independent of that coördinate, since then the group is essentially a group in two variables. There remain the two following cases.

Type VII. Same as II but with $\rho_1 = 1$.

$$x_1' = x_1, \qquad x_2' = mx_1 + x_2, \qquad x_3' = \rho_3^m x_3 \qquad (\rho_3 \neq 1).$$

Type VIII. Same as III but with $\rho = 1$.

$$x_1' = x_1,$$

$$x_2' = mx_1 + x_2,$$

$$x_3' = \frac{m(m-1)}{2!} x_1 + mx_2 + x_3.$$

The simplicity with which this enumeration is made may be compared with the determination of a complete list of these groups in terms of the infinitesimal transformation,* whence they may be obtained in finite form† by integration.

§ 8. *Real one-parameter groups.*—When m traverses its axis of reals, the point $(x_1', \cdots, x_n')$ describes a path curve. The group is called real, when every real point describes a real path curve. In other words, it is one which can be generated by a real infinitesimal transformation.

We have now to derive the necessary and sufficient conditions for the reality of the group corresponding to a real initial collineation C'. By the nature of our derivation of the group the question of its reality resolves itself into two parts. We have first to consider the reality of the reduction of C' to normal form, and after this the reality of the group in which C is contained. In the reduction we shall regard the transformation of C' through T as a change of variable.

When C' is real, the characteristic equation (2) is real and its elementary divisors, so far as they are imaginary, occur only in conjugate pairs of equal degree, $(\rho - \rho_i)^r$ and $(\rho - \bar{\rho}_i)^r$. It is fairly obvious, and can indeed be strictly verified, that the reduction of C' to normal form (1) can be made in

* Lie-Engel, loc. cit., vol. 3, p. 119, or Lie-Scheffers, loc. cit., p. 522.

† Cf. Fr. Meyer, loc. cit., p. 197–198.

such a way that to real elementary divisors will correspond real component substitutions (1), while to each conjugate pair of unreal elementary divisors will correspond two components (1) of equal order in which corresponding variables x_i, $\bar{x}_i$ of the two components are conjugate imaginaries. The variables of the real components (1) are real linear forms in terms of the initial variables of C', while the variables x_i, x_i of the conjugate components are linear forms with conjugate imaginary coefficients. We shall suppose henceforth that the reduction of C' to normal form C has been made in such a manner. Then C is not intrinsically imaginary, and to convert it into real form we have merely to place

$$x_i = y_i + iz_i, \qquad \rho = |\rho| \cdot [\cos\vartheta + i\sin\vartheta]$$

in one of each two conjugate components of (1), and then to split up the equations of (1) into their real and imaginary parts. When the same change is effected in (3), we obtain in place of a pair of conjugate imaginary components of order r the following equivalent set of $2r$ real equations:

$$\begin{aligned}
y_1' &= |\rho|^m \cdot [y_1 \cos m\vartheta - z_1 \sin m\vartheta],\\
z_1' &= |\rho^m| \cdot [y_1 \sin m\vartheta + z_1 \cos m\vartheta],\\
&\cdots\cdots\cdots\cdots\\
y_i' &= \frac{m(m-1)\cdots(m-i+1)}{(i-1)!}|\rho|^{m-i+1} \cdot [y_1 \cos(m-i+1)\vartheta\\
&\qquad - z_1 \sin(m-i+1)\vartheta]\\
&+ \frac{m(m-1)\cdots(m-i+2)}{(i-2!)}|\rho|^{m-i+2} \cdot [y_2 \cos(m-i+2)\vartheta\\
&\qquad - z_2 \sin(m-i+2)\vartheta] + \cdots + |\rho|^m \cdot [y_i \cos m\vartheta - z_i \sin m\vartheta]\\
z_i' &= \frac{m(m-1)\cdots(m-i+1)}{(i-1)!}|\rho|^{m-i+1} \cdot [y_1 \sin(m-i+1)\vartheta\\
&\qquad + z_1 \cos(m-i+1)\vartheta] + \cdots + |\rho|^m \cdot [y_i \sin m\vartheta + z_i \cos m\vartheta],\\
&\qquad (i = 2, \cdots, r).
\end{aligned} \tag{14}$$

These equations give in normal real form the component substitutions which correspond to a pair of conjugate elementary divisors of (2). By placing $m = 1$ we obtain the corresponding components of C itself.

We have thus ascertained the normal form into which a continuous one-parameter group containing a real substitution C' can be transformed *by real change of variable.* It consists, namely, only of components of the form (3) with real ρ and of components of the form (14). We have now to examine their reality when the parameter m is varied continuously through real values. The components (14) are plainly real. On the other hand, (3) will contain

an imaginary factor ρ^m if ρ is negative. In case, however, of the existence of a pair of equal elementary divisors of (2) with real ρ_i, we can regard them, if we please, as a pair of conjugate imaginary divisors $(\rho - \rho_i)^r$, $(\rho - \bar{\rho}_i)^r$ in which the common argument of ρ_i, $\bar{\rho}_i$ is 2π or π according as ρ_i is positive or negative. We may therefore employ, instead of two like components (3) of order r, a single system of $2r$ equations of form (14). If ρ_i is negative, we avoid thereby the introduction of two imaginary components into our group, while if ρ_i is positive, we have a choice between two different modes, both real, of generating the pair of like components of C. The following conclusion now ensues:

A necessary and sufficient condition that a real substitution C' can be included in a real, continuous and homogeneous one-parameter substitution group is that the negative roots of the characteristic equation (2) *shall occur in equal pairs of equal order. If there exist one or more pairs of equal positive roots of the same order, C' can be included in two or more real substitution groups of different nature.**

The reason for the failure of a real substitution C' to belong to a continuous real one-parameter group is to be sought in the inclusion of *projective reflections* in its make up. By a reflection on the axis or plane of x_i is to be understood a transformation $x_i' = -x_i$ which alters the sign of this coördinate, leaving the others unaltered. Suppose now that ρ in (1) is negative and set $\rho = -\rho'$. If we then make a change of coördinates by replacing each x_i, x_i' of even subscript by its negative, equations (1) become

$$\begin{aligned} -x_1' &= \rho' x_1, \\ -x_i' &= x_{i-1} + \rho' x_i \qquad (i = 2, \cdots, r), \end{aligned}$$

Clearly this transformation can be resolved into two which are commutative, the one being

$$\begin{aligned} x_1' &= \rho' x_1 \\ x_i' &= x_{i-1} + \rho' x_i \qquad (i = 2, \cdots, r). \end{aligned} \tag{15}$$

while the other consists of r reflections changing the signs of $x_1', \cdots, x_r'$. Now it is well known that an even but not an odd number of reflections can be generated by a real infinitesimal projective transformation. In fact, an even number of reflections $y_i' = -y_i$, $z_i = -z_i$ $(i = 1, \cdots, r)$ is included in a continuous group

$$\begin{aligned} y_i' &= y_i \cos m\pi - z_i \sin m\pi, \\ z_i' &= y_i \sin m\pi + z_i \cos m\pi \end{aligned} \qquad (i = 1, \cdots, r), \tag{16}$$

* *I. e.*, real groups differing in respect to the number or order of their components (3) and (14). If in constructing (14) the arguments of ρ, $\bar{\rho}$ are changed by 2π and -2π respectively, this changes the value of ϑ in (14) by 2π but does not change essentially the *nature* of the group.

which is commutative with (14). But we have found that a pair of components (15) of equal order and with equal ρ' can be included in a group (14) if we there put $\vartheta = 2\pi$, $|\rho| = \rho'$ and take y_i, z_i to be the r variables of the two components (15). As (14) and (16) are commutative, we see thus why the condition that the negative ρ_i shall occur only in equal pairs of equal orders is sufficient to insure the inclusion of C' in a real continuous group. We also saw above that this condition is necessary.

Consider next the reality of the group containing C' when C' is interpreted, not as a homogeneous linear substitution, but as a collineation in n homogeneous variables. A slight modification in the above results is now needed owing to the fact that we can suppress or supply a common factor in all the n variables x_i' of the collineation. The difficulty before arose from an imaginary factor ρ^m in the right-hand members of (3) when ρ was negative. But we are now at liberty to introduce a common factor $(-1)^m$ into all the n variables x_i'. If at the same time in each of the components (3) a change of coördinates is made by replacing every x_i', x_i of even subscript by its negative, we obtain the same equations as at start save for a change in the sign of every ρ. Thus the rôles of the positive and the negative roots of the characteristic equation are interchanged, while the argument of each imaginary root is changed by π. We reach therefore the following criterion for the reality of our group:

The necessary and sufficient condition that a real collineation C' in n homogeneous variables can be included in a continuous and real one-parameter collineation group or, in other words, that it can be generated by a real infinitesimal collineation, is that either the elementary divisors which correspond to negative roots of (2) *or those which correspond to positive roots shall occur only in equal pairs, or not at all.**

If this condition holds for the positive (negative) roots and there is at least one pair of equal negative (positive) roots of equal order, then C' can be included in two or more real groups of different nature, and only then.

The conclusion concerning the form of the real group which has been reached, though not expressly stated, can now be summed up as follows:

Every real and continuous one-parameter collineation group in n homogeneous variables, or homogeneous group in n variables, by proper choice of the coördinate planes can be expressed in a normal form which consists only of components of form (3), *in which the ρ_i are positive, and of components of form* (14).

§ 9. *Lists of real one-parameter groups.*—The method of §§ 6, 7 can be extended to construct lists of linear and homogeneous one-parameter groups under the added requirement that the groups shall be real. We have first

* This condition has been previously obtained by TABER who gave the condition in terms of the numbers of SYLVESTER. Cf. Bulletin of the American Mathematical Society, vol. 2 (1896), pp. 225–7, theorems 1, 2 and 6. The connection between these numbers and the degree of the elementary divisors is explained by him in vol. 3, pp. 157–8.

to throw together in all different ways sets of equations of form (3) and (14) for which the total number of equations shall be equal to n, and in those of form (3) ρ is now restricted to be positive. Then for each different combination of n equations all possible cases are to be distinguished which differ in respect to the invariantive configuration of points, lines, planes, etc., due account being taken of the homogeneous or non-homogeneous nature of the coördinates. For $n = 3$ we get seven types of real one-parameter collineation groups. Five of these are the same as in § 6, except that the ρ_i are now restricted to be positive. In addition, we have the two following types:

Type VI.
$$x_1' = \rho_1^m x_1,$$
$$y' = |\rho_2|^m \cdot [y \cos m\vartheta - z \sin m\vartheta],$$
$$z' = |\rho_2|^m \cdot [y \sin m\vartheta + z \cos m\vartheta] \qquad (\rho_2 = |\rho_2| e^{i\vartheta}),$$

in which $\rho_1 \neq |\rho_2|$.

Type VII. The same but with $\rho_1 = |\rho_2|$.*

A list for $n = 4$ is scattered through § 12, the total number of real types distinguished being 19. No account is there taken of distinctions similar to that just made between types VI and VII.

Of real homogeneous groups in three variables there are eleven types, eight of which have been already given in § 7, but the ρ_i therein are now taken to be positive. In addition, we have the types VI and VII just given with the added restriction $\rho_1 \neq 1$, $|\rho_2| \neq 1$, and an eleventh type which is the same as VI above but $|\rho_2| = 1$, $|\rho_1| \neq 1$. When $\rho_1 = 1$ in VI, the group is essentially one in two variables.

Part II. The Classification of Collineations. Space W-curves.

§ 10. *The mode of classification.*—The principle to be used for the classification of collineations is that of infinite iteration. As stated in the introduction, we assume at start the usual classification given by the theory of elementary divisors. The standard form there obtained for a collineation depends upon the nature of the system of elementary divisors of its characteristic equation, or, geometrically considered, upon the character of the invariantive configuration of points, lines, planes, etc. The equalities and inequalities subsisting between the roots ρ_i of the elementary divisors are thereby involved. When collineations of the same standard type are iterated, differences arise which depend upon *the relative magnitude of the moduli of the* ρ_i. Hence we get a further subdivision into *classes*. Although the effects of the iteration of a collineation and its inverse may be very different, it will be convenient never-

* For these groups in terms of the infinitesimal transformation see Lie-Scheffers, loc. cit., pp. 74–82.

theless to include both under the same class, inasmuch as they belong to a common group. Thereby the number of different classes is somewhat diminished.

The iterative principle applies alike to real and unreal collineations, demanding nothing beyond a knowledge of the characteristic equation. To effect the iteration of C we increase m through positive integral values in each component of C^m, while to iterate the inverse collineation m is to be decreased through negative integers. Instead, however, of changing m discontinuously we might allow it to increase or decrease continuously through real values. *Thus our classification is also simultaneously one for continuous one-parameter collineation groups.* It is, however, to be observed that if account is taken of reality, there is a very restricted class of cases (cf. § 8, near end) in which a real collineation can be included in two or more real collineation groups of different nature. To this extent the parallelism between the classification of collineations and of one-parameter collineation groups fails.

When m traverses its axis of reals starting from $m = 0$, the point $P' = (x_1', \cdots, x_n')$ starts from an initial position $P = (x_1, \cdots, x_n)$ and describes a path curve real or imaginary. These curves are sometimes called *W-curves* on account of their projective properties and are of especial interest when real. With the accompanying group they are then generated by real infinitesimal transformation. Thus our classification for real groups becomes one also for the system of path curves and is based essentially upon the character of this system.

An interesting extension of the notion of a W-curve will be obtained by allowing m to roam over its imaginary domain. Then the system of the components (3) and (14) of C^m is to be interpreted in a real space of $2n$-dimensions, in which P' traces a two-dimensional path locus. Into this extension I shall not enter here.

The classification of plane collineations ($n = 3$) brings out little essentially new, but will be developed briefly to illustrate the method and to facilitate the discussion for $n = 4$ which follows. It should, however, be pointed out that we are led *to distinguish* 9 *classes of one-parameter groups in a plane, as against* 7 *previously noted.* However, the two new classes (Nos. 5 and 8) are not real, even when the generating collineation ($m = 1$) is itself real.

§ 11. *Collineations in a plane* ($n = 3$).—I shall classify a plane collineation C simultaneously with the one-parameter group in which it is included. The equations of the former are obtained from those of the latter by putting $m = 1$. The five types of each which are ordinarily distinguished are repeated below in normal form. By iteration further subdivision into *classes* results, which will be numbered consecutively and independently of the type number under which they are found.

Type I. $x_i' = \rho_i^m x_i$, $(i = 1, 2, 3)$, $\rho_i \neq \rho_j$. Characterized by three invariant points and lines, which are taken to be the coördinate system. Assume $|\rho_1| \geqq |\rho_2| \geqq |\rho_3|$. Then on increasing and decreasing m either through integral values or continuously we obtain a subdivision into the following 3 classes.

Class 1. $|\rho_1| > |\rho_2| > |\rho_3|$.—When m starts from the value 0 and increases indefinitely, $P' = (x_1', x_2', x_3')$ leaves an initial position $P = (x_1, x_2, x_3)$ and approaches as a limit the invariant point $I_1 = (1, 0, 0)$, unless P should lie upon the invariant line $x_1 = 0$ when the limit is the second invariant point $I_2 = (0, 1, 0)$. For indefinite decrease of m the limit of P' is the third invariant point $I_3 = (0, 0, 1)$. When the ρ_i are all real and of like sign, C is generated by a real infinitesimal transformation and is said to be of hyperbolic type. The form of the path curves is well known.* A consideration of the ratios of x_1, x_1, x_3 for $m = +\infty$ or $-\infty$, will show that at I_1 and I_2 they are tangent to $x_3 = 0$ and $x_1 = 0$ respectively. When the ρ_i are not all of the same sign, C contains as a constituent a projective reflection, but the limiting effect of the iteration is not affected thereby.

Class 2. $|\rho_1| > |\rho_2| = |\rho_3|$.—No separate discussion is required for the assumption $|\rho_1| = |\rho_2| > |\rho_3|$, since the characteristic roots for our inverse substitution C^{-1} have the order of magnitude $|\rho_3^{-1}| = |\rho_2^{-1}| > |\rho_1^{-1}|$. For indefinite increase of m the point P' has the limit I_1, unless the initial point P lies upon $x_1 = 0$, while for indefinite decrease of m there is no limiting position of P'. Real groups of this class are obtained only when the initial transformation C' which has been transformed into C is real and gives a pair of conjugate imaginary roots ρ_2, ρ_3 for its characteristic equation (2). Then the real equations of the group have the form VI of § 9, where ρ_1 may be taken to be positive. The path curves take their simplest form when the line at infinity is chosen for the axis $x_1 = 0$, and when the two invariant points upon it are taken to be the circular points. Then $y = 0$, $z = 0$ are at right angles, and the path curves form a system of logarithmic spirals winding around the origin.

Class 3. $|\rho_1| = |\rho_2| = |\rho_3|$.—The indefinite increase or decrease of m gives no limiting position for P'. When C' and its group are real, the latter has the real form VII given in § 9. The logarithmic spirals of class 2 have now degenerated into concentric circles around the origin $y = 0$, $z = 0$.

Type II.

$$x_1' = \rho_1^m x_1,$$
$$x_2' = m\rho_1^{m-1} x_1 + \rho_1^m x_2, \qquad (\rho_1 \neq \rho_3).$$
$$x_3 = \rho_3^m x_3$$

Two invariant points, $I_2 = (0, 1, 0)$, $I_3 = (0, 0, 1)$.

* For drawings and further information concerning the plane W-curves, see SCHEFFERS, *Encyclopaedie der math. Wissenschaften*, III D4, §§ 13–19.

Class 4. $|\rho_1| > |\rho_3|$.—The indefinite increase and decrease of m give I_2 and I_3 respectively as the limits of P'. If ρ_1, ρ_3 are real and of same sign, the group is real. The system of path curves will be found in the second figure of SCHEFFERS, our equations showing that they are tangent to $x_3 = 0$ and $x_1 = 0$ at I_2 and I_3 respectively.

Class 5. $|\rho_1| = |\rho_3|$. In this case x_1', x_3' tend with increasing $|m|$ to become infinitesimal in comparison with x_2'. Hence whether m increases or decreases, P' has a unique limiting position I_2. An exception occurs if the initial point P lies upon the invariant line $x_1 = 0$, when there will be no limiting position for P'. There is no *real* collineation group of this class, since ρ_1, ρ_3 are of opposite sign when real. For $m = 1$, $\rho_1 = -\rho_3$, we have as the initial collineation C one which differs from the initial collineation of class 9 merely in the inclusion of a projective reflection. This prevents its generation by a real infinitesimal collineation.

Type III and *class* 6.
$$x_1' = \rho^m x_1,$$
$$x_2' = m\rho^{m-1} x_1 + \rho^m x_2,$$
$$x_3' = \frac{m(m-1)}{2!}\rho^{m-2} x_1 + m\rho^{m-1} x_2 + \rho^m x_3.$$

By cancellation of the common factor ρ^m in the three right-hand members, it will be seen that the path curves are rational, the form of the equations showing that they are conics which touch the only invariant line $x_1 = 0$ at the only invariant point $(0, 0, 1)$. They have, moreover, quadruple contact with one another in this point.

Types IV and V are degenerate cases of I and II in which the path curves are straight lines.

Type IV. Same as I but with $\rho_2 = \rho_3$.

IV, *Class* 7. $|\rho_1| > |\rho_2|$.—When m indefinitely increases or decreases, P' moves on a line $x_2 = cx_3$, and its two limiting positions are $(1, 0, 0)$ and the invariant intersection of this line with $x_1 = 0$.

Class 8. $|\rho_1| = |\rho_2|$.—There is no limiting position for P' on its path line. The group can not be real. C is a projective reflection when $\rho_1 = -\rho_2$.

Type V and *class* 9. Same as II but with $\rho_1 = \rho_3$. The limit of P' on the invariant line $x_1 = cx_3$ is the point $(0, 1, 0)$ irrespective of whether m increases or decreases.

In conclusion, it may be pointed out that the collineations of classes 7 and 8, namely
$$x_1' = \rho_1 x_1, \qquad x_2' = \rho_2 x_2, \qquad x_3' = \rho_2 x_3,$$
can also be included in groups of class 2 or 3 respectively. By placing them,

as has been done, in groups characterized by exactly the same invariant points rather than in groups with fewer such points, an exact parallelism between the classification of collineations and one-parameter collineation groups has been attained. When collineations of class 7 and 8 are real, they can, of course, be included in real groups of type VI and VII, § 9.

§ 12. Collineations in three dimensions ($n = 4$).

I shall, as before, classify simultaneously collineations and one-parameter collineation groups. The conditions characterizing types and classes will be given in their general form, but *the discussion will be confined to the real representatives of the classes. Consequently throughout the discussion the ρ_i are assumed to be positive or pairs of conjugate imaginaries.* The principal aim in view is to find the real groups which give real W-curves. The possible types of real groups are denoted by G_i ($i = 1, 2, \cdots$). By the iterative principle the 13 types of space collineations and one-parameter groups subdivide into 34 classes. In the last 16 of these classes the path curves lie in planes passing through a common axis, and in two others (11 and 17) the path curves are imaginary. *I obtain therefore* 16 *real types of space W-curves.**

When invariant quadric surfaces can be formed under any class, this fact is noted.

Type I. $\qquad x_i' = \rho_i^m x_i, \qquad \rho_i \neq \rho_j \qquad (i = 1, 2, 3, 4).$

It may be assumed that $|\rho_1| \geqq |\rho_2| \geqq |\rho_3| \geqq |\rho_4|$.

Class 1. $|\rho_1| > |\rho_2| > |\rho_3| > |\rho_4|$.—This case is parallel to case 1 of plane collineations. In each of the four invariant planes $x_i = 0$ we obtain plane collineations. When $P' = (x_1', \cdots, x_4')$ starts from any initial position $P = (x_1, \cdots, x_4)$ outside of these planes, it describes a true space curve terminating for $m = +\infty$ in the invariant point $I_1 = (1, 0, 0, 0)$ and for $m = -\infty$ in $I_4 = (0, 0, 0, 1)$. This particular class of W-curves is very well known and has the important property that their tangents cut the coördinate planes in four points whose anharmonic ratio is constant, not merely for a single path curve but for the entire system.† Consideration of ratios of the x_i' shows that the path curves at I_1 are tangent to the line $x_3 = 0$, $x_4 = 0$ and have $x_4 = 0$ for their osculating plane. A similar result holds at I_4.

* Scheffers (III D.IV, p. 214) seems to say that the number of varieties is 13. Although this is the number of types of continuous one-parameter collineation groups when no account is taken of reality, several varieties or *classes* of W-curves are to be found under some of these types.

† Cf. Klein and Lie, Comptes Rendus, loc. cit. Other interesting properties are given. Their attention is confined for the most part to the curves belonging to class 1. Cf. also Scheffers, loc. cit., § 20.

When $\rho_1 \rho_4 = \rho_2 \rho_3$, the path curves lie upon the invariant hyperboloids $x_1 x_4 = \text{cst.}\, x_2 x_3$. Also if either ρ_2^2 or ρ_3^2 is equal to the product of two of the other three ρ_i, the curves will lie upon a surface of second order. If both $\rho_1 \rho_4 = \rho_2 \rho_3$ and $\rho_2^2 = \rho_1 \rho_3$, then ρ_3^m / ρ_4^m may be taken as a parameter τ, in terms of which the equations of the path curves are

$$x' = \tau^3 x, \qquad y' = \tau^2 y, \qquad z' = \tau z,$$

where

$$x = x_1 / x_4, \qquad y = x_2 / x_4, \qquad z = x_3 / x_4.$$

Hence for this special case we have a system of twisted cubics, having contact in two of the invariant points.

Class 2. $|\rho_1| > |\rho_2| = |\rho_3| > |\rho_4|$.—To obtain real path curves we must suppose ρ_2, ρ_3 to be conjugate imaginaries and employ correspondingly two equations of form (14). We obtain thus a second type of *real* one-parameter group:

$$(G_2) \qquad \begin{aligned} x_i' &= \rho_i^m x_i, \\ x_j' &= \rho_j^m x_j \qquad (\rho_i \neq \rho_j), \\ y_1' &= \sigma^m [y_1 \cos m\vartheta - z_1 \sin m\vartheta], \\ z_1' &= \sigma^m [y_1 \sin m\vartheta + z_1 \cos m\vartheta], \end{aligned}$$

where in the particular case before us we have

$$i = 1, \quad j = 4, \quad \rho_i = \rho_1, \quad \rho_j = \rho_4, \quad \sigma e^{i\vartheta} = \rho_3, \quad \sigma e^{-i\vartheta} = \rho_2.$$

Here real coördinates y_1, z_1 are used in place of $x_2 = y_1 + iz_1$, $x_3 = y_1 - iz_1$. The form of the equations shows that as m traverses its axis of reals positively or negatively, all path curves outside of the invariant planes tend toward the two real invariant points $I_1 = (1,0,0,0)$ and $I_4 = (0,0,0,1)$ respectively as poles, and wind an infinite number of times around the joining line.

To study the curves further take $x_1 = 0$ to be the plane at infinity and use ordinary orthogonal coördinates

$$x = \frac{x_j}{x_i}, \qquad y = \frac{y_1}{x_i}, \qquad z = \frac{z_1}{x_i}.$$

Then the path curves will lie upon the surfaces of revolution

$$x^2 = \text{cst.}\, (y^2 + z^2)^k, \tag{17}$$

where

$$k = \frac{\log \rho_j / \rho_i}{\log \sigma / \rho_i}.$$

This is generated by a plane W-curve $x = cr^{k/2}$ of class 1. Upon these surfaces of revolution the path curves wind around the x-axis in such a manner

that their orthogonal projections upon $x = 0$ are logarithmic spirals having the origin as pole.

A special case of interest is that in which $\sigma^2 = \rho_1 \rho_4$. Then the surfaces of revolution are paraboloids generated by the parabolas $x = cr^2$. We can collineate these into ellipsoids and hyperboloids having double contact with one another. If, in fact, we put (17) into the homogeneous form $x_1 x_4 = \text{cst.}\,[y_1^2 + z_1^2]$ and choose for $x_1 = 0$ and $x_4 = 0$ two parallel planes cut orthogonally by $y_1 = 0, z_1 = 0$ in the points I_4 and I_1 respectively, this will be accomplished. The ellipsoids of revolution lie between the parallel planes which they touch in I_4 and I_1, and among them is one which is a sphere. It can now be shown quickly that the W-curves upon the sphere are spherical loxodromes. To prove this, it suffices to show that seen from I_4 they are the stereographic projections upon the sphere of a set of logarithmic spirals which lie in the plane $x_4 = 0$ tangent to the sphere at the diametrically opposite point I_1 and have this point as their common pole. Now the equations of the cones which project the W-curves from I_4 as center are obtained from G_2 by omitting the equation $x_4' = \rho_4^m x_4$. Consider the section of this cone with $x_4 = 0$. If the homogeneous coördinates receive the usual interpretation by which they are proportional to the distances of the point from the corresponding coördinate planes, y_1' and z_1' for any point of the section will be proportional to the distances of the point from the axis of y_1 and z_1 in this plane. The equations for y_1', z_1' show then that the section is a logarithmic spiral with the origin I_1 as pole. Hence the W-curves upon our family of ellipsoids and hyperboloids are their intersections with cones having these logarithmic spirals for their bases and I_4 for their common vertex. In particular, their intersections with the sphere are the desired loxodromes. This special case of a W-curve has been previously noted.*

Class 3.† $|\rho_1| > |\rho_2| > |\rho_3| = |\rho_4|$.—The real groups of this class have the above form G_2 but with the special values

$$i = 1, \quad j = 2, \quad \rho_3 = \sigma e^{i\vartheta}, \quad \rho_4 = \sigma e^{-i\vartheta}.$$

The discussion is quite similar to that for class 2. In the generating curve (17) we now have $k < 1$, whereas in class 2 we have $k > 1$. As before, the W-curves are spirals winding around the axes of x and having I_1 as a pole, but the surfaces of revolution (17) upon which the spirals lie so spread away

* Cf. KLEIN and LIE, loc. cit., p. 1224.

† This case is identical with that for which $|\rho_1| = |\rho_2| > |\rho_3| > |\rho_4|$, since for the inverse substitution in class 3 the characteristic roots have the reverse order $|\rho_4^{-1}| = |\rho_3^{-1}| > |\rho_2^{-1}| > |\rho_1^{-1}|$. A corresponding remark applies to the other classes, and the reader will find that all possibilities have been duly cared for.

from the x-axis as not to pass through I_4. This point is no longer a second asymptotic pole of the spirals.

Class 4. $|\rho_1| > |\rho_2| = |\rho_3| = |\rho_4|$.—We have the same equations as for class 3 but with $k = 1$. Hence the surfaces (17) are cones of revolution. The W-spirals upon these cones are doubly loxodromic, cutting at one constant angle the pencil of planes through the axis of the cone and also at a constant angle a second pencil of planes parallel to $x = 0$.* The invariance of the two angles follows a consequence that the orthogonal differentials dx', $dr \equiv d\sqrt{(y')^2 + (z')^2}$ and $(y'\,dz' - z'\,dy')/\sqrt{(y')^2 + (z')^2}$, due to any infinitesimal change in the parameter m, contain a common factor ρ_j^m/ρ_i^m, which is the only part dependent upon m.

Classes 5 and 6. $|\rho_1| = |\rho_2| > |\rho_3| = |\rho_4|$.—To obtain a real group we must take ρ_1, ρ_2 and ρ_3, ρ_4 to be pairs of conjugate imaginaries. We introduce accordingly two sets of equations (14) and obtain the following type of real one-parameter group:

$$(G_3)\qquad \begin{aligned} y_1' &= \sigma_1^m [y_1 \cos m\vartheta_1 - z_1 \sin m\vartheta_1], \\ z_1' &= \sigma_1^m [y_1 \sin m\vartheta_1 + z_1 \cos m\vartheta_1] && (\rho_1 = \sigma_1 e^{i\vartheta_1}), \\ y_2' &= \sigma_2^m [y_2 \cos m\vartheta_2 - z_2 \sin m\vartheta_2], \\ z_2' &= \sigma_2^m [y_2 \sin m\vartheta_2 + z \cos m\vartheta_2] && (\rho_2 = \sigma_2 e^{i\vartheta_2}). \end{aligned}$$

The four invariant points are now pairs of conjugate imaginaries which lie upon the two opposite invariant edges, $y_1 = 0, z_1 = 0$ and $y_2 = 0, z_2 = 0$, of our tetraedron of reference. Obviously P' lies in two planes revolving around these edges as axes. If, for convenience, these edges are taken perpendicular to each other and also the coördinate planes which meet in each, the two moving planes change uniformly with m, revolving at rates depending upon ϑ_1 and ϑ_2. At the same time the ratio of the distances P' from the two axes, $\sqrt{(y_1')^2 + (z_1')^2}/\sqrt{(y_2')^2 + (z_2')^2}$, increase in geometric ratio as σ_1^m/σ_2^m. While it may be said that in general the path curves wind simultaneously around the two edges, the statement may require proper modification because of the passage of P' to the plane at infinity during a revolution of either plane.

The characteristics of the path curves depend upon the relative commensurability or incommensurability of ϑ_1 and ϑ_2. We consequently distinguish the two following cases.

* SCHEFFERS, loc. cit., p. 252. Cf. also LIE-SCHEFFERS, *Vorlesungen über Differentialgleichungen*, p. 244.

Class 5. $\sigma_1 > \sigma_2$; ϑ_2 *and* ϑ_2 *relatively commensurable.*—Suppose $q\vartheta_2 = p\vartheta_1$ where p and q are integers relatively prime. After a change of $q\pi / \vartheta_1$ in the value of m the two revolving planes return to their initial positions, having performed q and p half revolutions respectively about their axes. Hence the path-spiral cuts an infinite number of times every intersection of the two revolving planes.

A special case of interest is that for which $\vartheta_1 = \pm \vartheta_2$. Because the two planes revolve at the same rate, their intersections compose one set of generators of a ruled surface of second order. The axes of revolution belong to the other set and divide the surface into two regions. The generating point P' can not cross these axes since they are invariant lines for G_3 and must therefore remain continually in one of the two regions. This can only be if P' passes to infinity during each half-revolution of the two planes. The W-curve which it traces therefore traverses the region from infinity to infinity an infinite number of times. Furthermore, as m indefinitely increases or decreases, σ_1^m / σ_2^m tends toward ∞ or 0, and $\sqrt{(y_2')^2 + (z_2')^2}$ tends to become infinitesimal in comparison with $\sqrt{(y_1^2) + (z_1^2)}$ or *vice versa.* The branches of the W-curve tend to approximate in the one case to the axis $y_2 = 0$, $z_2 = 0$ and in the other case to $y_1 = 0$, $z_1 = 0$.

Class 6. *Same as class* 5 *but with* ϑ_1, ϑ_2 *relatively incommensurable.*—The path curve meets each intersection of the two revolving planes but once.

Class 7. $\sigma_1 = \sigma_2$; ϑ_1, ϑ_2 *unequal and relatively commensurable.*—This case differs from case 5, in that P' returns to its initial position after an increment of $\pm q\pi / \vartheta_1$ or $\pm 2q\pi / \vartheta_1$ in m according as $q - p$ is even or odd. Consequently the path curves now close. They may be regarded as degenerate cases of the unclosed spirals of class 5, just as the circles of class 3 of plane collineations have been looked upon as degenerate cases of the logarithmic spirals of the preceding case.

Let r denote the larger of the two numbers $|p|$ and $|q|$. Then if all the right hand members of G_3 be multiplied by $\sec^r m\vartheta_1 / q$, they can be expressed as polynomials in terms of $\tau = \tan m\vartheta_1 / q$ or $\tau = \tan m\vartheta_1 / 2q$ according as $q - p$ is even or odd. It thus appears that the *W-curves of this class are rational curves of degrees* r *and* $2r$ *respectively.* Furthermore each curve lies upon a ruled invariant quadric

$$(y_1')^2 + (z_1')^2 = k^2 [(y_2')^2 + (z_2')^2],$$

where k has the value $\sqrt{y_1^2 + z_1^2} / \sqrt{y_2^2 + z_2^2}$ depending upon the initial point $P = (y_1, z_1, y_2, z_2)$. Two particular generators of this quadric belonging to opposite systems have the equations

$$\frac{y_1'}{\sqrt{y_1^2+z_1^2}}-\frac{y_2'}{\sqrt{y_2^2+z_2^2}}=0, \qquad \frac{z_1'}{\sqrt{y_1^2+z_1^2}}-\frac{z_2'}{\sqrt{y_2^2+z_2^2}}=0.$$

and

$$\frac{y_1'}{\sqrt{y_1^2+z_1^2}}-\frac{y_2'}{\sqrt{y_2^2+z_2^2}}=0, \qquad \frac{z_1'}{\sqrt{y_1^2+z_1^2}}+\frac{z_2'}{\sqrt{y_2^2+z_2^2}}=0.$$

Substituting here the values for $y_1', \cdots, z_2'$ given in G_3 and putting

$$\sin\alpha_1=\frac{y_1}{\sqrt{y_1^2+z_1^2}}, \qquad \cos\alpha_1=\frac{z_1}{\sqrt{y_1^2+z_1^2}}, \qquad \sin\alpha_2=\frac{y_2}{\sqrt{y_2^2+z_2^2}},$$

we obtain the following pairs of equations for determining the intersections of the curve with the two generators:

$$\sin(m\vartheta_1-\alpha_1)=\sin(m\vartheta_2-\alpha_2), \qquad \cos(m\vartheta_1-\alpha_1)=\cos(m\vartheta_2-\alpha_2),$$

and

$$\sin m(\vartheta_1-\alpha_1)=\sin(m\vartheta_2-\alpha_2), \qquad \cos(m\vartheta_1-\alpha_1)=-\cos(m\vartheta_2-\alpha_2).$$

These have respectively the solutions

$$\frac{m\vartheta_1}{q}=\frac{\alpha_1-\alpha_2}{q-p}+\frac{2n\pi}{q-p} \qquad (n=0,\ 1,\ 2,\ \cdots)$$

and

$$\frac{m\vartheta_1}{q}=\frac{\pi+\alpha_1+\alpha_2}{q+p}+\frac{2n\pi}{q+p}.$$

When $q-p$ is given, there are $|q-p|/2$ of the former solutions and $|q+p|/2$ of the latter which lie within an interval of π for $m\vartheta_1/q$. Hence the curves then intersect one system of generators of the quadric in $|q-p|/2$ points and the other in $|q+p|/2$ points. When $q-p$ is odd, these numbers are to be doubled.

Class 8. $\sigma_1=\sigma_2$; ϑ_1 *and* ϑ_2 *unequal and relatively incommensurable.*— The curves lie upon the ruled quadrics of class 7, but they are no longer rational and they cut each generator an infinite number of times.

Type II.

$$(G_4)\qquad \begin{aligned} x_1' &= \rho_1^m x_1, \\ x_2' &= m\rho_1^{m-1}x_1+\rho_1^m x_2, \\ x_3' &= \rho_3^m x_3, \\ x_4' &= \rho_4^m x_4 \end{aligned} \qquad (\rho_1+\rho_3+\rho_4).$$

There are three invariant points: $I_1 = (0, 1, 0, 0)$, $I_2 = (0, 0, 1, 0)$, $I_3 = (0, 0, 0, 1)$. When $|m|$ increases indefinitely, x_1 becomes infinitesimal in comparison with x_2'. The nature of the path curves can best be seen by using the rectangular coördinates $x = x_2/x_1$, $y = x_3/x_1$, $z = x_4/x_1$. The equations of the curves become

$$x' = \frac{m}{\rho_1} + x, \qquad y_1' = \left(\frac{\rho_3}{\rho_1}\right)^m y, \qquad z' = \left(\frac{\rho_4}{\rho_1}\right)^m z. \tag{18}$$

The point P' traces a W-curve on a cylinder whose axis is parallel to the x-axis and whose intersection with the (y, z)-plane is a plane W-curve of class 1. The position of these W-curves relatively to the invariant points depends upon the inequalities subsisting between the moduli of the ρ_i.

Class 9. $|\rho_1| > |\rho_3| > |\rho_4|$.—The path curves run between I_1 and I_3, having at I_1 a common tangent line, $x_3 = 0$, $x_4 = 0$, and a common osculating plane $x_4 = 0$ while at I_3 they are tangent to the opposite edge of the coördinate tetraedron and have $x_1 = 0$ as their common osculating plane. When $\rho_3^2 = \rho_1 \rho_4$, the curves lie upon the invariant quadric surfaces $x_3^2 = cx_1 x_4$.

Class 10. $|\rho_3| > |\rho_1| > |\rho_4|$. The path curves run between I_2 and I_3, having at each point an invariant coördinate axis as a common tangent and an invariant coördinate plane for osculating plane. When $\rho_1^2 = \rho_3 \rho_4$, they lie upon the quadric $x_1^2 = cx_3 x_4$.

Class 11. $|\rho_1| = |\rho_3| > |\rho_4|$. The group and path curves are imaginary.

Class 12. $|\rho_1| \neq |\rho_3| = |\rho_4|$.—Taking ρ_3, ρ_4 to be conjugate imaginaries, we have the following fifth type of real one-parameter group:

$$(G_5) \qquad \begin{aligned} x_1' &= \rho_1^m x_1, \\ x_2' &= m\rho_1^{m-1} x_1 + \rho_1^m x_2, \\ y_1' &= \sigma^m [y_1 \cos m\vartheta - z_1 \sin m\vartheta], \\ z_1' &= \sigma^m [y \sin m\vartheta + z_1 \cos m\vartheta] \end{aligned} \qquad (\sigma e^{i\vartheta} = \rho_3)$$

If we use the rectangular coördinates $x = x_2/x_1$, $y = y_2/x_1$, $z = z_2/x_1$, then in place of the last two equations of (18) we have two equations similar to the last two of G_5. Hence the path curves now lie upon cylinders whose cross sections are logarithmic spirals (i. e., plane W-curves of class 2).

Class 13. $|\rho_1| = |\rho_3| = |\rho_4|$.—Since $\rho_1 = \sigma$, a common factor σ^m can be removed from all four equations of G_5. The logarithmic cross sections of

the cylinders of the preceding case become concentric circles. All the W-curves are ordinary helices with a common axis.*

Type III.

$$(G_6)\qquad \begin{aligned} x_1' &= \rho_1^m x_1, & x_3' &= \rho_3^m x_3, \\ x_2' &= m\rho_1^{m-1} x_1 + \rho_1^m x_2, & x_4' &= m\rho_3^{m-1} x_3 + \rho_3^m x_4 \end{aligned} \qquad (\rho_3 \neq \rho_1).$$

Using the same rectangular coördinates as under type II, we have in place of the last two equations for y', z' in (18) two others which are similar to those just given for x_3', x_4'. The cross section of our cylinder is therefore now a plane W-curve of class 4.

Class 14. $|\rho_1| \neq |\rho_3|$.—The path curves run between the only two invariant points $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$, where they touch two invariant lines; etc.

Class 15. $|\rho_1| = |\rho_3|$.—Taking ρ_1, ρ_3 to be conjugate imaginaries we get the following seventh type of real group:

$$(G_7)\qquad \begin{aligned} y_1' &= \sigma^m [y_1 \cos m\vartheta - z_1 \sin m\vartheta], \qquad z_1' = \sigma^m [y_1 \sin m\vartheta + z_1 \cos m\vartheta], \\ y_2' &= m\sigma^{m-1} [y_1 \cos (m-1)\vartheta - z_1 \sin (m-1)\vartheta] \\ &\qquad + \sigma^m [y_2 \cos m\vartheta - z_2 \sin m\vartheta], \\ z_2' &= m\sigma^{m-1} [y_1 \sin (m-1)\vartheta + z_1 \cos (m-1)\vartheta] \\ &\qquad + \sigma^m [y_2 \sin m\vartheta + z_2 \cos m\vartheta]. \end{aligned}$$

For convenience, take the non-intersecting axes $y_1 = 0$, $z_1 = 0$ and $y_2 = 0$, $z_2 = 0$ to be perpendicular. Then P' lies in two planes revolving about these axes. One of these planes turns uniformly about the former axis, for uniform change in m, but the motion of the other plane depends on two groups of terms of rotary character in y_2', z_2'. One of these groups is affected with a multiplier m, and hence as $|m|$ increases indefinitely, the other group becomes relatively infinitesimal. In consequence, the motion of the second plane about its axis tends to become uniform. Now an increment of π/ϑ in m will change simultaneously the signs of all four coördinates of P' and for excessively great values of $|m|$ alters the position of P' only infinitesimally. But it also turns the first plane through a half revolution about its axis $y_1 = 0$, $z = 0$. As P' can not cross this invariant axis and returns very nearly to its initial position after the half rotation, it must have passed out to infinity in the

* Cf. SCHEFFERS, loc. cit., p. 215.

course of the half rotation. Furthermore, as $|m|$ increases indefinitely, y_1', z' tend to become infinitesimal in comparison with $\sqrt{(y_2')^2+(z_2')^2}$. We see therefore that our path curve consists of an infinity of pieces which with indefinitely increasing or decreasing m tend to approximate to the only invariant real line of the collineation, $y_1 = 0, z_1 = 0$.

If $\rho_1 = -\rho_3$ our path curves are imaginary but lie upon invariant quadrics, $x_2 x_3 + x_1 x_4 + c x_1 x_3 = 0$.

Type IV.

$$(G_8)\qquad \begin{aligned} x_1' &= \rho_1^m x_1, \\ x_2' &= m\rho_1^{m-1} x_1 + \rho_1^m x_2, \\ x_3' &= \frac{m(m-1)}{2}\rho_1^{m-2} x_1 + m\rho_1^{m-1} x_2 + \rho_1^m x_3, \\ x_4' &= \rho_4^m x_4 \qquad (\rho_4 \neq \rho_1). \end{aligned}$$

Class 16. $|\rho_1| \neq |\rho_4|$.—The path curves connect the only two invariant points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, at each of which they touch one of the two invariant lines and osculate one of the two invariant planes. If rectangular coördinates are introduced as under type II, they will lie upon cylinders whose cross sections are conics (cf. class 6 of plane collineations).

Class 17. $|\rho_1| = |\rho_4|$.—The path curves are imaginary, and have the point $(0, 0, 1, 0)$ as terminus or limiting point, irrespective of whether m increases or decreases indefinitely.

Type V and *class* 18.

$$(G_9)\qquad \begin{aligned} x_1' &= \rho^m x_1, \\ x_2' &= m\rho^{m-1} x_1 + \rho^m x_2, \\ x_3' &= \frac{m(m-1)}{2!}\rho^{m-2} x_1 + m\rho^{m-1} x_2 + \rho^m x_3, \\ x_4' &= \frac{m(m-1)(m-2)}{3!}\rho^{m-3} x_1 + \frac{m(m-1)}{2!}\rho^{m-2} x_2 + m\rho^{m-1} x_3 + \rho^m x_4. \end{aligned}$$

After cancellation of the common factor ρ^m we have polynomials in m of degree $\leqq 3$, which shows that the W-curves are twisted cubics. They pass through the only invariant point where they touch the only invariant straight line and osculate the only invariant plane. As in type IV, the curves lie upon invariant cones defined by the first three equations.

This completes the enumeration of the cases in which the W-curves are true space curves. In the remaining types of collineations two or more of the ρ_i of the component collineations are equal. If $\rho_i = \rho_j$, we have a pair of equations of form $x_i = \rho_i^m x_i$, $x_j' = \rho_j^m x_j$, so that the path curves must lie in the invariant planes of the pencil $x_i = cx_j$. In the remaining classes the systems of path curves differ from one another in respect to the disposition and types of the plane W-curves involved. The classes will be enumerated without further comment.

Type VI. Same as I with $a\rho_i$ and ρ_j equal.

Class 19. $\rho_1 = \rho_2$; $|\rho_1| > |\rho_3| > |\rho_4|$.

Class 20. $\rho_2 = \rho_3$; $|\rho_1| > |\rho_2| > |\rho_3|$.

Class 21. $\rho_1 = \rho_2$; $|\rho_1| = |\rho_3| > |\rho_4|$.

Class 22. $\rho_1 = \rho_2$; $|\rho_1| \neq |\rho_3| = |\rho_4|$.—This gives an additional type of real group, the same as G_2 but with $\rho_i = \rho_j$.

Class 23. $\rho_1 = \rho_2$; $|\rho_1| = |\rho_3| = |\rho_4|$.

Type VII. Same as I but with $\rho_1 = \rho_2 = \rho_3$.

Class 24. $|\rho_1| \neq |\rho_4|$.

Class 25. $|\rho_1| = |\rho_4|$.

Type *VIII.* Same as I but with $\rho_1 = \rho_2 \neq \rho_3 = \rho_4$.

Class 26. $|\rho_1| \neq |\rho_3|$.

Class 27. $|\rho_1| = |\rho_3|$.—This gives another type of real group, the same as G_3 but with $\sigma_1 e^{i\phi_1} = \sigma_2 e^{i\phi_2}$.

Type IX. Same as II but with $\rho_1 = \rho_3$.

Class 28. $|\rho_1| \neq |\rho_4|$.

Class 29. $|\rho_1| = |\rho_4|$.

Type X. Same as II but with $\rho_3 = \rho_4$.

Class 30. $|\rho_1| \neq |\rho_3|$.

Class 31. $|\rho_1| = |\rho_3|$.

Type XI and *class* 32. Same as II but with $\rho_1 = \rho_3 = \rho_4$.

Type XII and *class* 33. Same as III but with $\rho_1 = \rho_3$.

Type XIII and *class* 34. Same as IV but with $\rho_1 = \rho_4$.

These 8 types furnish 10 types of real groups, making 19 in all.

§13. *The associated quadratic complexes.*—It is well known that in the case of the one-parameter group of type I, §11, the tangents to its path curves form a quadratic complex, commonly called the *tetraedral complex.* The six line coördinates of the tangent to the path curve through $(x_1', \cdots, x_4')$ are evidently given by expressions of the form

$$p_{ij} \equiv \begin{vmatrix} x_i' & x_j' \\ x_i' + dx_i' & x_j' + dx_j \end{vmatrix} = \begin{vmatrix} x_i' & x_j' \\ dx_i' & dx_j' \end{vmatrix} = \begin{vmatrix} \rho_i^m x_i & \rho_j^m x_j \\ \rho_i^m x_i \log \rho_i & \rho_j^m x_j \log \rho_j \end{vmatrix}$$

$$= x_i x_j \rho_i^m \rho_j^m \log \frac{\rho_j}{\rho_i}.$$

They accordingly satisfy the quadratic equation

(I) $$A p_{12} p_{34} + B p_{13} p_{24} + C p_{14} p_{23} = 0,$$

when the coefficients are connected by the relation

$$A \log \frac{\rho_2}{\rho_1} \log \frac{\rho_4}{\rho_3} + B \log \frac{\rho_3}{\rho_1} \log \frac{\rho_4}{\rho_2} + C \log \frac{\rho_4}{\rho_1} \log \frac{\rho_3}{\rho_2} = 0.$$

The coördinates of the tangent lines of the path curves for types II–V can be found in similar manner. They furnish quadratic complexes of the following types:

II. $$A p_{13} p_{14} + B p_{12} p_{34} = 0,$$

where

$$A \log \frac{\rho_3}{\rho_1} \log \frac{\rho_4}{\rho_1} + \frac{B}{\rho} \log \frac{\rho_4}{\rho_3} = 0.$$

III. $$A p_{13}^2 + B p_{12} p_{34} = 0,$$

where

$$A \left[\log \frac{\rho_3}{\rho_1} \right]^2 + \frac{B}{\rho_1 \rho_3} = 0.$$

IV. $$A (p_{12} p_{24} - p_{13} p_{14}) + B p_{12} p_{14} = 0,$$

where

$$A \left(-\frac{1}{\rho_1} + \frac{1}{2\rho_1} \log \frac{\rho_4}{\rho_1} \right) + B \log \frac{\rho_4}{\rho_1} = 0.$$

V. $$p_{12} (p_{23} + p_{14}) - p_{13}^2 - \frac{1}{12\rho^2} p_{12}^2 = 0.$$

The apparent form of these equations can be modified by means of the identity $p_{12} p_{34} + p_{13} p_{42} + p_{14} p_{23} = 0$. Each equation and the identity together constitute a pair of quadratic forms whose elementary divisors serve to classify the complex. The characteristics of the five types will be seen to be respectively [(11) (11) (11)], [(22) (11)], [(211) (11)], [(33)], [(321)]. The "singular surface" of the complex for Type I consists of the four invariant coördinate planes. This invariant tetraedron degenerates in the four succeeding types, but the singular surface comprises those of the coördinate planes which are invariant.

For types VI–XIII the tangents to the path curves form a *special linear* complex, inasmuch as the path curves lie in a pencil of planes and accordingly their tangents meet the axis of the pencil. We conclude therefore that *there are just 5 types of quadratic complexes which consist of the tangents to the path curves of a continuous one-parameter collineation group.* As under our first 5 types of one parameter groups we distinguished various classes of *real* groups, so under our 5 types of quadratic complexes we may distinguish corresponding classes of *real* complexes, whose real equations are easily found.

MADISON, WISCONSIN.

BICOMBINANTS OF THE RATIONAL PLANE QUARTIC AND COMBINANT CURVES OF THE RATIONAL PLANE QUINTIC*

BY

J. E. ROWE

Introduction.

Each covariant locus of a rational plane curve contributes its share toward the covariant and invariant theory of rational curves of higher order by reason of the fact that osculants† of rational curves play the same rôle in regard to rational curves as polars do in regard to binary forms. Especially is this true of covariant points and covariant lines of the rational curve of order n (which we shall call R^n) because corresponding to each covariant line or point of an R^n is a series of covariant rational curves of any R^k, where $k > n$; the parametric equations of these covariant rational curves are easy to obtain. In the first section of this paper a new kind of covariant locus is defined for R^n and illustrated by important loci associated with the R^4. Section II is devoted to combinant curves of the R^5. The theory of combinant curves of the R^5 in the plane corresponds to the invariant theory of the R^5 in space; this correspondence is pointed out and illustrated. Finally a sufficient number of combinants and combinative m-ics of two line sections of the plane R^5 have been found to form the algebraic basis of an analytic treatment of combinant curves of the R^5 in the plane, which is shown to be identical with the invariant theory of the R^5 in space.

Section I. Bicombinants of the Rational Plane Quartic Curve.

Bicombinants.

Combinant curves of R^n have already been defined in these Transactions‡ as covariant loci of R^n which are combinants of two line sections of the R^n. Covariant loci of a slightly different kind are considered in this section, and as frequent reference to the above article is necessary, the letters used there to designate curves or expressions are employed with the same meaning in this

* Presented to the Society, September 12, 1911.

† See THOMSEN, American Journal of Mathematics, vol. 32 (1910).

‡ *Important Covariant Curves and a Complete System of Invariants of the Rational Quartic Curve*, these Transactions, vol. 12 (1911), pp. 295–310.

present article. Also the results obtained there will be used here often without specific reference.

Let the R^4 be written parametrically

$$x_i = a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i \equiv f_i^4; \tag{1}$$

and the R^n

$$x_i = a_i t^n + nb_i t^{n-1} \cdots \equiv f_i^n. \qquad (i = 0, 1, 2). \tag{2}$$

The combinant curves of an R^n are expressible in terms of three-rowed determinants of the type

$$|abx| = \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix} \tag{3}$$

by means of a translation scheme* already explained in detail and to be used also in the second section of this paper in considering a set of covariant curves of the R^5. The combinants of the three binary forms f_i are expressible in terms of three-rowed determinants of the type

$$|abc| \equiv \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}. \tag{4}$$

There is a class of covariant curves of the R^n whose equations are composed of terms which are combinations of determinants of the types (3) and (4). Evidently these are not purely combinant curves as already defined, and inasmuch as such curves in form partake of the nature of the combinants of three binary forms as well as of the nature of combinant curves, they will be referred to as Bicombinant Curves, or Bicombinants, of the R^n. It may happen that a combinant curve and an invariant of the R^n together form a bicombinant; thus combinant curves are special cases of bicombinants. Such functions may be derived as follows: Let the parameters of a set of covariant points on an R^n be given by the binary form $(\gamma t)^n = 0$, whose coefficients are combinations of determinants of the type (4); also let a combinative m-ic of two line sections be $(\delta t)^m = 0$, whose coefficients are combinations of determinants of the type (3); each of these binary forms is unaltered by a linear transformation of the x's; hence any simultaneous invariant* of $(\gamma t)^n = 0$ and $(\delta t)^m = 0$ is a covariant of the R^n and possesses the properties peculiar to bicombinants. We shall now apply this to the R^4.

For convenience let

* Evidently any such function is unaltered by a linear transformation of the t's.

$$(5)\quad \begin{cases} \alpha = |\,abx\,|, & \beta = |\,adx\,|, & \lambda = |\,acx\,|, & \mu = |\,bcx\,|, & \gamma = |\,aex\,|, \\ \alpha' = |\,dex\,|, & \beta' = |\,bex\,|, & \lambda' = |\,cex\,|, & \mu' = |\,cdx\,|, & \delta = |\,bdx\,|. \end{cases}$$

Among these ten Greek letters the following identities exist:

$$(6)\quad \begin{cases} \beta\mu - \lambda\delta + \alpha\mu' = 0, & \beta'\mu' - \lambda'\delta + \alpha'\mu = 0, \\ \alpha\lambda' - \beta'\lambda + \gamma\mu = 0, & \alpha'\lambda - \beta\lambda' + \gamma\mu' = 0, \\ \alpha\alpha' + \gamma\delta - \beta\beta' = 0. \end{cases}$$

The expression

$$(7)\qquad (\beta - 3\mu)\,t^2 + (\gamma - 2\delta)\,t + (\beta' - 3\mu') = 0$$

is a combinative quadratic of two line sections of the R^4. The osculant cubic of the R^4 at a point t' is an R^3 with three flexes on a line; if the equation of this flex line is formed and t' made variable, the result is (7). Hence for a given value of t, say t_1, (7) becomes the equation of the flex line of the cubic osculant at t_1; for the coördinates of a point (7) becomes a quadratic whose roots are the names of the two cubic osculants whose flex lines pass through the given point. These flex lines of cubic osculants envelope a conic which is given parametrically by (7). The discriminant of (7) is the point equation of this conic which in our notation* is called $A - 12B = 0$. The conic on the flexes of the R^4 cuts out two other points which shall be called q_i. These are given by the quadratic

$$(8)\quad [\,2\,|\,acd\,| - |\,abc\,|\,]t^2 + [\,8\,|\,bcd\,| - |\,ace\,|\,]t + [\,2\,|\,bce\,| - |\,ade\,|\,] = 0;$$

for convenience let this be written

$$(9)\qquad Q_0 t^2 + Q_1 t + Q_2 = 0.$$

The condition for (7) and (9) to be apolar is

$$(10)\qquad L_1 = 2Q_0(\beta' - 3\mu') + 2Q_2(\beta - 3\mu) + (2\delta - \gamma)\,Q_1 = 0,$$

a covariant line of the R^4, the locus of points such that tangents from any point of it to the conic $A - 12B = 0$ are apolar to the quadratic (8), i. e., L_1 cuts out (8) on the conic $A - 12B = 0$.

The R^4 has six flexes whose parameters are roots of the sextic

$$(11)\quad \begin{aligned} &|\,abc\,|\,t^6 + 2\,|\,abd\,|\,t^5 + [\,3\,|\,acd\,| + |\,abc\,|\,]t^4 + [\,2\,|\,ace\,| + 4\,|\,bcd\,|\,]t^3 \\ &\qquad + [\,3\,|\,bce\,| + |\,ade\,|\,]t^2 + 2\,|\,bde\,|\,t + |\,cde\,| = 0. \end{aligned}$$

* L. c., p. 1.

A combinative sextic of two line sections is

$$\alpha t^6 + 3\lambda t^5 + 3[\beta + 2\mu]t^4 + [\gamma + 8\delta]t^3 + 3[\beta' + 2\mu']t^2 + 3\lambda' t + \alpha' = 0: \tag{12}$$

for a given t, say t_1, (12) becomes the equation of the tangent to the R^4 at that point; for the coördinates of a point it yields six roots which are the parameters of the six tangents from the given point to the R^4. The apolarity condition of (11) and (12) yields a second covariant line which if reduced by identities of the type

$$|abc|\alpha' - |abd|\lambda' = |bcd|\gamma - |acd|\beta' \tag{13}$$

becomes the same as L_1. Hence, *the locus of points such that tangents drawn from it to the R^4 are apolar to the flexes is the line L_1, which cuts out the q_i on $A - 12B = 0$.*

• Another set of covariants which can be expressed as bicombinants is illustrated by the following example. The two tangents to $A - 12B = 0$ at the two points (9) may be found by substituting t_1 and t_2 in (7) for t, multiplying the resulting expressions, and replacing the symmetric functions of t_1 and t_2 by the proper coefficients of (9). Calling the product of these tangents Q' (evidently they are the flex lines of the cubic osculants of the points q_i) we find

$$\begin{aligned} Q' = {} & [\beta^2 - 6\beta\mu + 9\mu^2]Q_2^2 + [\beta'^2 - 6\beta'\mu' + 9\mu'^2]Q_0^2 + [\gamma^2 - 4\gamma\delta + 4\delta^2]Q_0Q_2 \\ & - [\beta\gamma - 2\beta\delta - 3\mu\gamma + 6\mu\delta]Q_1Q_2 - [\beta'\gamma - 2\beta'\delta - 3\mu'\gamma - 6\mu'\delta]Q_0Q_1 \\ & + [\beta\beta' - 3\beta\mu' - 3\beta'\mu + 9\mu\mu'][Q_1^2 - 2Q_0Q_2] = 0. \end{aligned} \tag{14}$$

The following relation exists:

$$L_1^2 - 4Q' = [Q_1^2 - 4Q_0Q_2][A - 12B]; \tag{15}$$

this is simply the conic $A - 12B = 0$ expressed in terms of two tangents and the chord of contact; $Q_1^2 - 4Q_0Q_2 = 0$ is the condition for the two points q_i to unite. If equation (12) had been used instead of (9), by the same process the equation of the two tangents to the R^4 at the points q_i could have been obtained; in this manner many important covariants of R^n can be expressed in bicombinant form.

The apolarity condition of (9) cubed and (12) gives a covariant line L_3, the locus of points such that tangents from it to the R^4 are apolar to the points q_i each taken three times. It is evident from geometrical considerations that the two q_i are on this line when $Q_1^2 - 4Q_0Q_2 = 0$. It will be shown that the covariant line of the R^4 which is on the q_i, and two other points which we shall call r_i, is a member of the pencil

$$KL_1 + L_3 = 0, \tag{16}$$

where L_3 calculated from (9) and (12) is

$$\begin{aligned} L_3 = {} & 20[Q_0^3\alpha' + Q_2^3\alpha] - 30[Q_0^2Q_1\lambda' + Q_2^2Q_1\lambda] \\ & + 12[Q_0Q_1^2 + Q_0^2Q_2][\beta' + 2\mu'] + 12[Q_2Q_1^2 + Q_0Q_2^2][\beta + 2\mu] \\ & - Q_1[Q_1^2 + 6Q_0Q_2][\gamma + 8\delta] = 0. \end{aligned} \tag{17}$$

The methods used in finding the equation of the line determined by the q_i are the same as those used to prove the two following theorems in regard to the R^4 in a new way. Two combinant curves of the R^4 are:

$$A = 12\lambda\lambda' - 48\mu\mu' + \gamma^2 + 16\delta^2 - 8\alpha\alpha' - 8\beta\beta' = 0, \tag{18}$$

which is the conic touching the six flex tangents of the R^4; and

$$B = \lambda\lambda' - \mu\mu' - \beta\mu' - \beta'\mu + \delta^2 - \alpha\alpha' = 0, \tag{19}$$

which is the equation of the conic described by the intersections of the flex tangents of cubic osculants of the R^4. The parameters of the six flexes of the R^4 are given by equation (11), a binary sextic which is a simultaneous covariant of the two binary quartics* which symmetrically represent the R^4, or which form a pencil of binary quartics—the fundamental involution of the R^4. Every covariant† of a binary form, or every simultaneous covariant of two binary forms can be obtained when its seminvariant, or leading coefficient, is given. Equation (11) is really the jacobian of the two quartics just referred to. Any member of the pencil of conics

$$A + K'B = 0 \tag{20}$$

cuts out a set of covariant parameters on the R^4, and each octavic so obtained is a covariant of the two quartics of the fundamental involution of the R^4. Hence, *if any member of the pencil* (20) *can be found which cuts the* R^4 *in eight parameters given by a binary octavic whose leading coefficient is divisible by* $|abc|$, *the leading coefficient of* (11), *then equation* (11) *must be a factor of this octavic and the flexes of the* R^4 *must be on the corresponding conic of* (20). Evidently any term in A or B involving

$$\alpha, \quad \beta, \quad \gamma, \quad \text{or} \quad \lambda \tag{21}$$

cannot contribute a term towards the coefficient of t^8 of the binary octavic. Neglecting the terms of A and B which involve the letters (21), we have

* W. F. MEYER, *Apolarität und rationale Curven*, p. 9.

† SALMON, *Higher Algebra*, Fourth Edition, Lesson XIII, 13.

$$A = -48\mu\mu' + 16\delta^2, \tag{22}$$

$$B = -\mu\mu' - \beta'\mu + \delta^2. \tag{23}$$

The combination

$$A - 16B = 0 \tag{24}$$

is a member of the pencil (20) cutting out of R^4 eight points given by a binary octavic whose leading coefficient* is divisible by $|abc|$. Hence (24) is the conic on the flexes of the R^4. Incidentally, this amounts to a proof of the fact that they do lie on a conic. Also, *if without imposing a condition on the curve we find that one of a covariant set of points lies on a particular covariant curve, all of this set lie on the curve.* If the parameters 0 and ∞ are assigned to the points of contact of a double tangent, the equations of the R^4 may be made to assume the form

$$\begin{cases} x_0 = a_0t^4 + 4b_0t^3 + 6c_0t^2 + 4d_0t, \\ x_1 = \quad 4b_1t^3 + 6c_1t^2 + 4d_1t + e_1, \\ x_2 = \quad 6c_2t^2. \end{cases} \tag{25}$$

Calculating the values of the determinants (5) for the point (0 1 0), we find

$$A = -48\mu\mu', \qquad B = -\mu\mu'. \tag{26}$$

The conic

$$A - 48B = 0 \tag{27}$$

is on one point of contact of a double tangent and must be on all such contacts.

These two proofs illustrate the methods to be used in finding the equation of the covariant line on the q_i and the process employed in the second section in finding the equation of the combinant cubic on the flexes of the R^5.

Applying what we have just stated to the pencil of lines (16), we find that L_4, the covariant line of the R^4 on the q_i, is defined as follows:

$$L_3 + 4[Q_1^2 - 4Q_0Q_2]L_1 = 5L_4. \tag{28}$$

Hence,

$$\begin{aligned} L_4 = 4[Q_0^3\alpha' + Q_2^2\alpha] - 6[Q_0^2Q_1\lambda' + Q_2^2Q_1\lambda] + [4Q_0Q_1^2 - 4Q_0^2Q_2]\beta' \\ + [4Q_2Q_1^2 - 4Q_0Q_2^2]\beta + 24Q_0^2Q_2\mu' + 24Q_0Q_2^2\mu - 16Q_0Q_1Q_2\delta \\ + [2Q_0Q_1Q_2 - Q_1^3]\gamma = 0. \end{aligned} \tag{29}$$

If the parametric equations of the R^4 are substituted in (29), and (9) factored out, we find the two r_i are given by the quadratic

* By the same reasoning it may be shown that the two q_i are on the conic (24).

$$
\begin{aligned}
&[4Q_0^2\,|\,ade\,|+6Q_0Q_1\,|\,ace\,|+4\,(Q_1^2-Q_0Q_2)\,|\,abe\,|+24Q_2^2\,|\,abc\,|+24Q_0Q_2\,|\,acd\,|\\
&\quad-16Q_1Q_2\,|\,abd\,|\,]t^2+[\,16Q_0^2\,|\,bde\,|+16Q_2^2\,|\,abd\,|-24Q_0Q_1\,|\,bce\,|\\
&\quad-24Q_1Q_2\,|\,acd\,|-4Q_1Q_2\,|\,abe\,|-4Q_0Q_1\,|\,ade\,|+96Q_0Q_2\,|\,bcd\,|\\
&\quad+6Q_1\,|\,ace\,|\,]t+[\,4Q_2^2\,|\,abe\,|-6Q_2Q_1\,|\,ace\,|+4\,(Q_1^2-Q_0Q_2)\,|\,ade\,|\\
&\quad+24Q_0^2\,|\,cde\,|+24Q_2Q_0\,|\,bce\,|-16Q_0Q_i\,|\,bde\,|\,]=0\,.
\end{aligned}
\tag{30}
$$

The condition for (9) and (30) to be harmonic is C_1', i. e., the condition for four collinear flexes. Hence, when the conic $A - 16B = 0$ degenerates,* four flexes are on one line and the two remaining flexes and the two q_i on the other, *the latter two flexes and q_i being harmonic*, since a simple substitution shows that in this case the r_i coincide with the two flexes on the line with the q_i.

A Covariant Point of the R^4.

The intersection of L_1 and L_3 is a covariant point P of the R^4, which possesses the following properties: (1) the six tangents from P to R^4 have their parameters apolar to the flex parameters and to the parameters of the q_i each taken three times; (2) the line cutting out the two q's on R^4 passes through P; (3) the line cutting out the two q_i on the conic $A - 12B = 0$ passes through P.

The parameters of the two points on $A - 12B = 0$ whose tangents to $A - 12B = 0$ meet at P can be found by solving (10) and (17) for x_0, x_1, and x_2 and substituting these values in equations (7). The result is a quadratic in t whose roots are the required parameters. The coördinates of P are of degree 16 in the coefficients of the parametric equations of the R^4; hence the quadratic just mentioned has coefficients of degree six in the three-rowed determinants of the type $|\,abc\,|$. Since this quadratic is apolar to (8), the line cutting out the parameters which are its roots on $A - 12B = 0$ passes through the intersection of the tangents to $A - 12B = 0$ at the two points q_i. If we recall that the q_i and the flexes of the R^4 lie on a conic $(A-16B=0)$, and that the flex parameters are given by the jacobian of two binary quartics whose third transvectant yields the parameters of the q_i, the properties found for P become more interesting because they suggest the possibility of constructing six points which are related to six other points and to two associated points in the same way that the six points of contact of tangents from P to R^4 are related to the six flexes and the two q_i.

* See A. Brill, Mathematische Annalen, vol. 17 (1880). In this article Brill shows that if an R^4 has three flexes on a line, it must have four on the line.

Bicombinant Conics.

Adjoint curves of R^n are curves which pass through the nodes and certain other points necessary to account for their complete intersection with the R^n. For instance the adjoint conic of the R^4 cuts in six nodal parameters and in two other points, say S_1 and T_1; if S_1 and T_1 are definite points then this curve is known as the adjoint conic of the R^4 through the points S_1 and T_1, or simply as the adjoint of S_1 and T_1; if the conic cuts out the nodes and touches the R^4 at S_1, the conic is called the adjoint of S_1. An adjoint cubic of the R^4 cuts out the nodal parameters and six other points. Hence for every pair of points on the R^4 there is a definite adjoint conic; for every set of six points there is a definite adjoint cubic. In general such curves are not covariants of the R^4, but if the points other than the nodes which are cut out of the R^4 by an adjoint conic or cubic happen to be a set of covariant parameters on the R^4, this adjoint is certainly a covariant curve of the R^4 as it cuts the R^4 in only covariant points. Stahl* has indicated how the adjoint curves of orders $n-1$ and $n-2$ of an R^n can be written down as determinants of order $3n-2$, although he says nothing of covariant adjoints. There is a definite conic on the nodes and the points q_i; a second conic on the nodes and the points r_i, and a definite conic on the nodes and any covariant set of points given by a quadratic. The adjoint of the q_i will be written out in bicombinant form, and any other may be obtained by substituting instead of Q_0, Q_1, and Q_2 the corresponding coefficients of any other quadratic whose roots are a pair of covariant parameters.

The bicombinant conic on the nodes and the points q_i of the R^4 of equation (1) is

$$
\begin{aligned}
&Q_0\left\{\begin{array}{r}
24\,|\,abc\,|\,(36\lambda'^2-96\alpha'\mu'-16\alpha'\beta') \\
-16\,|\,abd\,|\,(24\beta'\lambda'-64\alpha'\delta-4\alpha'\gamma) \\
+4\,|\,abe\,|\,(16\beta'^2-96\alpha'\mu-6\lambda'\gamma)+24\,|\,acd\,|\,(6\lambda'\gamma-16\alpha'\beta) \\
-6\,|\,ace\,|\,(4\beta'\gamma-24\alpha'\lambda)+4\,|\,ade\,|\,(\gamma^2-16\alpha\alpha')
\end{array}\right\} \\
(32)\quad &+Q_1\left\{\begin{array}{r}
96\,(\,|\,abc\,|\,\alpha'\gamma+|\,cde\,|\,\alpha\gamma)-256\,(\,|\,abd\,|\,\alpha'\beta+|\,bde\,|\,\alpha\beta') \\
-16\,(\,|\,abe\,|\,\beta'\gamma+|\,ade\,|\,\beta\gamma)+96\,(\,|\,abe\,|\,\beta\lambda'+|\,ade\,|\,\beta'\lambda) \\
+576\,(\,|\,acd\,|\,\alpha'\lambda+|\,bce\,|\,\alpha\lambda')+6\,|\,ace\,|\,(\gamma^2-36\lambda\lambda')
\end{array}\right\} \\
&+Q_2\left\{\begin{array}{r}
24\,|\,cde\,|\,(36\lambda^2-96\alpha\mu-16\alpha\beta)-16\,|\,bde\,|\,(24\beta\lambda-64\alpha\delta-4\alpha\gamma) \\
+4\,|\,ade\,|\,(16\beta^2-96\alpha\mu'-6\lambda\gamma)+24\,|\,bce\,|\,(6\lambda\gamma-16\alpha\beta') \\
-6\,|\,ace\,|\,(4\beta\gamma-24\alpha\lambda')+4\,|\,abe\,|\,(\gamma^2-16\alpha\alpha')
\end{array}\right\}=0.
\end{aligned}
$$

*Mathematische Annalen, vol. 38 (1891), pp. 583–4.

Bicombinant Cubics.

Corresponding to each set of six covariant parameters on the R^4 is a definite covariant cubic cutting out these and the nodes from R^4. Hence the cubic on the nodes and flexes of the R^4 can be found in this way. Also the adjoint cubic of the nodes themselves is simply the equation of the three lines forming the natural reference triangle of the R^4. This cubic cuts out of the R^4 the nodal sextic counted twice. We shall write out the equation of the cubic on the nodes and flexes of the R^4, understanding that the six flex parameters are given by

$$F_0 t^6 + F_1 t^5 + F_2 t^4 + F_3 t^3 + F_4 t^2 + F_5 t + F_6 = 0. \tag{33}$$

Its equation is

$$\begin{cases} F_0\begin{pmatrix} 4\alpha'\gamma^2 + 64\alpha'\gamma\delta - 48\beta'\lambda'\gamma - 768\alpha'\beta'\mu + 64\beta'^3 + 144\gamma\lambda'\mu' + 864\lambda'^2\mu \\ - 384\beta'\lambda'\delta + 1024\alpha'\delta^2 - 64\alpha\alpha'^2 - 384\alpha'\mu'\beta + 144\alpha'\lambda\lambda' - 2304\alpha'\mu\mu' \end{pmatrix} \\ + F_1\begin{pmatrix} 96\alpha'\beta' + 576\alpha'\mu'\lambda - 216\lambda\lambda'^2 - 16\alpha'\beta\gamma - 256\alpha'\beta\delta + 96\beta\beta'\lambda \\ + 6\gamma^2\lambda' + 96\alpha'\gamma\mu - 16\beta'^2\gamma \end{pmatrix} \\ + F_2\begin{pmatrix} - 64\alpha\alpha'\beta' - 384\alpha\alpha'\mu' + 144\alpha\lambda'^2 + 64\alpha'\beta^2 - 24\beta\gamma\lambda' \\ - 24\alpha'\gamma\lambda + 4\beta'\gamma^2) \end{pmatrix} \\ + F_3\begin{pmatrix} (32\alpha\alpha'\gamma + 256\alpha\alpha'\delta - 96\alpha\beta'\lambda' - 96\alpha'\beta\lambda + 36\lambda\lambda'\gamma - \gamma^3) \\ + F_4^*(\quad) \quad + F_5(\quad) \quad + F_6(\quad) = 0; \end{pmatrix} \end{cases} \tag{34}$$

evidently the covariant combinant cubic on the nodes and any set of six covariant parameters can be obtained by substituting the coefficients of the sextic yielding these covariant parameters for the F's in (34). Equation (30) squared multiplied by (9) gives a sextic; if the coefficients of this sextic be substituted in (34) for the F's, the result is a covariant bicombinant cubic cutting out the nodes, q_i, and touching the R^4 at the points r_i; the cube of equation (9) substituted for (33) makes (34) a covariant bicombinant cubic on the nodes, osculating the R^4 at each point q.

Section II. Combinant Curves of the Rational Plane Quintic and Their Relation to Invariants of the Rational Quintic Curve in Space.

Let the R^5 be written parametrically

$$x_i = a_i t^5 + 5b_i t^4 + 10c_i t^3 + 10d_i t^2 + 5e_i t + f_i \quad (i = 0, 1, 2). \tag{1}$$

* Coefficients of F_4, F_5, and F_6 are obtained by exchanging primed and unprimed letters in F_2, F_1, and F_0.

Cutting (1) by two lines

$$(\xi x) = 0 \quad \text{and} \quad (\eta x) = 0 \tag{2}$$

we obtain the two binary quintics

$$(\alpha t)^5 \equiv (a\xi)\, t^5 + 5\,(b\xi)\, t^4 + 10\,(c\xi)\, t^3 \cdots = 0, \tag{3}$$

$$(\beta t)^5 \equiv (a\eta)\, t^5 + 5\,(b\eta)\, t^4 \cdots = 0. \tag{4}$$

The combinants of (3) and (4) are expressible in terms of two-rowed determinants of the matrix

$$\left\| \begin{matrix} (a\xi) & (b\xi) & (c\xi) & (d\xi) & (e\xi) & (f\xi) \\ (a\eta) & (b\eta) & (c\eta) & (d\eta) & (e\eta) & (f\eta) \end{matrix} \right\|. \tag{5}$$

If x_0, x_1, and x_2 are substituted for the coördinates of the point in which the lines (2) intersect, any two-rowed determinant of (5) becomes a three-rowed determinant; for instance,

$$\begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} \text{ becomes } \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix} \equiv |abx| \tag{6}$$

by this process. Hence following out this scheme of interpretation, we have Covariant Curves of the R^5 corresponding to combinants of (3) and (4).

The following abbreviations will be found convenient:

$$\begin{cases} \alpha = |abx|, & \beta = |acx|, & \gamma = |adx|, & \delta = |aex|, & \lambda = |bcx|, \\ \alpha' = |efx|, & \beta' = |dfx|, & \gamma' = |cfx|, & \delta' = |bfx|, & \lambda' = |dex|, \\ \mu = |bdx|, & \mu' = |cex|, & \psi = |afx|, & \varphi = |bex|, & \chi = |cdx|. \end{cases} \tag{7}$$

Among these Greek letters exist the identities

$$\begin{cases} \alpha\alpha' - \delta\delta' + \psi\varphi = 0, \quad \beta\beta' - \gamma\gamma' + \psi\chi = 0, \quad \gamma\gamma' - \mu\mu' + \varphi\chi = 0. \\ \alpha\chi - \beta\mu + \gamma\lambda = 0, \quad \alpha'\chi' - \beta'\mu' + \gamma'\lambda' = 0, \\ \alpha\mu' - \beta\varphi + \delta\lambda = 0, \quad \alpha'\mu - \beta'\varphi + \delta'\lambda' = 0, \\ \alpha\gamma' - \beta\delta' + \psi\lambda = 0, \quad \alpha'\gamma - \beta'\delta + \psi\lambda' = 0, \\ \alpha\lambda' - \gamma\varphi + \mu\delta = 0, \quad \alpha'\lambda - \gamma'\varphi + \mu'\delta' = 0, \\ \alpha\beta' - \gamma\delta' + \psi\mu = 0, \quad \alpha'\beta - \gamma'\delta + \psi\mu' = 0, \\ \beta\lambda' - \gamma\mu' + \delta\chi = 0, \quad \beta'\lambda - \gamma'\mu + \delta'\chi = 0. \end{cases} \tag{8}$$

The first transvectant of (3) and (4) or their jacobian is

$$\begin{aligned}(9)\qquad & \alpha t^8 + 4\beta t^7 + (6\gamma + 10\lambda)\, t^6 + (4\delta + 20\mu)\, t^5 + (\psi + 15\varphi + 20\chi)\, t^4 \\ & + (4\delta' + 20\mu')\, t^3 + (6\gamma' + 10\lambda')\, t^2 + 4\beta' t + \alpha' = 0;\end{aligned}$$

for a given t this becomes the equation of the tangent to the R^5 at that point; for the coördinates of a point it yields the parameters of the eight tangents to the R^5 from the given point.

The third transvectant of (3) and (4) is

$$\begin{aligned}(10)\qquad & (\gamma - 3\lambda)\, t^4 + (2\delta - 4\mu)\, t^3 + (\psi + \varphi - 8\chi)\, t^2 \\ & + (2\delta' - 4\mu')\, t + (\gamma' - 3\lambda') = 0:\end{aligned}$$

the osculant cubic of the R^5 has three flexes on a line, and if t_1 is substituted in (10) for t the result is the flex line of the cubic osculant at the point t_1; for the coördinates of a point (10) yields the parameters of four points, the flex lines of whose cubic osculants are on the given point. These flex lines of cubic osculants envelope a rational class curve of order four which shall be referred to as the ρ^4 of the R^5. From (10) the parametric equations of this ρ^4 are

$$\begin{aligned}(11)\qquad \xi_i = {} & [(a_u d_v) - 3(b_u c_v)]\, t^4 + [2(a_u e_v) - 4(b_u d_v)]\, t^3 \\ & + [(a_u f_v) + (b_u e_v) - 8(c_u d_v)]\, t^2 + [2(b_u f_v) - 4(c_u e_v)]\, t \\ & + [(c_u f_v) - 3(d_u e_v)] \qquad (i, u, v = 0, 1, 2).\end{aligned}$$

This curve is of the fourth class and sixth order and is therefore a rational sextic with six cusps.

The fifth transvectant of (3) and (4) is

$$(12)\qquad P = \psi - 5\varphi + 10\chi = 0 = 5P';$$

this is the well-known covariant line of the R^5 which cuts out a set of the fundamental involution.

Twelve times the g_2 of (10) we call Q; hence

$$\begin{aligned}(13)\qquad Q = {} & 12\gamma\gamma' - 36(\gamma\lambda')^* + 108\lambda\lambda' - 12\delta\delta' + 24(\mu\delta') - 48\mu\mu' \\ & + (\psi + \varphi - 8\chi)^2 = 0;\end{aligned}$$

this is the conic on the cusps of the ρ^4.

One hundred and forty times the g_2 of (9) we call T, and we find that

$$(14)\qquad 3T = 28P^2 - 25Q;$$

evidently any two members of this pencil of conics may be so combined as to form the covariant line squared, hence *this pencil of conics has double contact and the covariant line P is on the two points of contact.*

* Such an expression as $\gamma\lambda'$ when in parenthesis carries with it the conjugate expression $\gamma'\lambda$.

Two hundred and sixteen times the g_3 of (10) we call D_3, which written in full is

$$\begin{aligned} D_3 = {} & (\psi + \varphi - 8\chi)[36\gamma\gamma' - 108(\gamma\lambda') + 324\lambda\lambda' + 18\delta\delta' \\ & - 36(\mu\delta') + 72\mu\mu'] - 54(\gamma\delta'^2) + 216(\mu'\delta'\gamma) - 216(\mu'^2\gamma) \\ & + 162(\lambda\delta'^2) - 648(\mu'\delta'\lambda) + 648(\lambda\mu'^2) - (\psi + \varphi - 8\chi)^3 = 0. \end{aligned} \tag{15}$$

Calculating an invariant of the third order in the coefficients for a binary octavic written in the usual way with binomial coefficients, I find

$$\begin{aligned} I_3 = {} & \alpha_0\alpha_4\alpha_8 - 4(\alpha_0\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_8) + 3(\alpha_0\alpha_6^2 + \alpha_2^2\alpha_8) \\ & - 8(a_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_7) + 12\alpha_1\alpha_4\alpha_7 - 22\alpha_2\alpha_4\alpha_6 \\ & + 24(\alpha_2\alpha_5^2 + a_3^2\alpha_6) - 36\alpha_3\alpha_4\alpha_5 + 15\alpha_4^3 = 0. \end{aligned} \tag{16}$$

Every binary octavic has a covariant octavic of degree two in its coefficients, which yields eight points whose fourth polars as to the octavic are self-apolar sets. The apolarity condition of this covariant and the original octavic is a multiple of the above invariant. Applying what has been stated to (9), we find that if $14^3 \cdot 25I_3$ is called $3H$,

$$\begin{aligned} 3H = {} & (5\psi + 75\varphi + 100\chi)[196\alpha\alpha' + 588\beta\beta' - 198\gamma\gamma' - 330(\gamma\lambda') \\ & - 550\lambda\lambda' - 36\delta\delta' - 180(\mu\delta') - 900\mu\mu'] - 9{,}800(\alpha\beta'\delta') \\ & - 49{,}000(\alpha\beta'\mu') + 9{,}450(\alpha\gamma'^2) + 31{,}500(\alpha\gamma'\lambda') \\ & + 26{,}250(\alpha\lambda'^2) - 4{,}200(\beta\delta'\gamma') - 7{,}000(\beta\lambda'\delta') \\ & - 21{,}000(\beta\mu'\gamma') - 35{,}000(\beta\mu'\lambda') + 1{,}800(\gamma\delta'^2) \\ & + 18{,}000(\gamma\mu'\delta') + 45{,}000(\gamma\mu'^2) + 3{,}000(\lambda\delta'^2) \\ & + 30{,}000(\mu'\delta'\lambda) + 75{,}000(\lambda\mu'^2) + 3(\psi + 15\varphi + 20\chi)^3 = 0. \end{aligned} \tag{17}$$

The quartic osculant of the R^5 at a point t' is

$$\begin{aligned} x_i = {} & (a_i t' + b_i) t^4 + 4(b_i t' + c_i) t^3 + 6(c_i t' + d_i) t^2 \\ & + 4(d_i t' + e_i) t + (e_i t' + f_i) \quad (i = 0, 1, 2). \end{aligned} \tag{18}$$

This is an R^4, and if the conic A as described in Section I is formed for (18), the result will involve t' to the fourth power; making t' variable we obtain

$$
(19)\quad \left\{\begin{aligned}
&[12\beta\mu' - 48\lambda\chi + \delta^2 + 16\mu^2 - 8\gamma\varphi - 8\alpha\lambda']\,t^4 + [12\beta\gamma' + 4\beta\lambda' \\
&\quad + 4\gamma\mu' - 36\lambda\mu' - 16\mu\chi + 24\mu\varphi + 2\delta\psi + 6\delta\varphi - 8\alpha\beta' - 8\gamma\delta']\,t^3 \\
&\quad + [4\beta\beta' + 4\gamma\gamma' - 44\lambda\lambda' + 12\,(\gamma\lambda') - 32\chi^2 - 24\mu\mu' - 6\delta\delta' \\
&\quad + \psi^2 + 2\psi\varphi + 9\varphi^2 + 32\varphi\chi - 8\alpha\alpha' - 8\,(\mu\delta')]\,t^2 + [12\beta'\gamma \\
&\quad + 4\beta'\lambda + 4\gamma'\mu - 36\lambda'\mu - 16\mu'\chi + 24\mu'\varphi + 2\delta'\psi - 6\delta'\varphi \\
&\quad - 8\alpha'\beta - 8\gamma'\delta']\,t + [12\beta'\mu - 48\lambda'\chi + \delta'^2 + 16\mu'^2 - 8\gamma'\varphi \\
&\qquad\qquad - 8\alpha'\lambda'] = 0;
\end{aligned}\right.
$$

for a given t, (19) becomes the equation of the conic on the flex tangents of the osculant quartic at that point; for the coördinates of a point, (19) becomes a quartic whose roots are the parameters of four osculant quartics whose conics A are on the given point. Likewise, by using the conic B we obtain from (18) by the same process

$$
(20)\quad \left\{\begin{aligned}
&[\beta\mu' - \lambda\chi - \gamma\chi - \varphi\lambda + \mu^2 - \alpha\lambda']\,t^4 + [\beta\gamma' - \mu'\lambda + \varphi\mu - \delta\chi \\
&\quad - \lambda\delta' - \alpha\beta']\,t^3 + [\gamma\gamma' - \alpha\alpha' - \lambda\lambda' - \mu\mu' - (\mu\delta') + \varphi^2]\,t^2 \\
&\quad + [\beta'\gamma - \mu\lambda' + \varphi\mu' - \delta'\chi - \lambda'\delta - \alpha'\beta']\,t + [\beta'\mu - \lambda'\chi \\
&\qquad\qquad - \gamma'\chi - \varphi\lambda' + \mu'^2 - \alpha'\lambda'] = 0;
\end{aligned}\right.
$$

for a given t this becomes the conic B for the osculant quartic at that point; for the coördinates of a point, (20) becomes a quartic whose roots are the parameters of four osculant quartics whose conics B are on the given point. Both (19) and (20) are combinative quartics of two line sections of the R^5. Six times the apolarity condition of (10) and (19) we call A_3, and six times the apolarity condition of (10) and (20) we call B_3. These combinant cubics are expressible in terms of P, Q, D_3, and H, which are not connected by any linear relation. We find that

$$
(21)\qquad 46{,}250B_3 = 1{,}375D_3 - 81H + 4{,}250PQ - 2{,}744P^3,
$$

and

$$
(22)\qquad 7{,}875A_3 = 125D_3 - 162H + 13{,}650PQ - 5{,}488P^3.
$$

It is much easier to work with A_3 and B_3 than with D_3 and H, hence other combinant cubics will be expressed in terms of A_3, B_3, P, and Q.

Every binary octavic has four points whose second polars as to itself form self-apolar sets. These points are given by a quartic covariant of degree two in the coefficients of the octavic. Forming this covariant for (9) and requiring it to be apolar to (10), the resulting condition is proportional to

$$27PQ - 170B_3 - 30A_3 = 0. \tag{23}$$

If equation (9) is required to be apolar to the square of (10), the condition may be expressed as

$$130B_3 - 5A_3 + 8PQ = 0. \tag{24}$$

Of course, we know from the theory of the binary quartic that the condition for (10) and its Hessian to be apolar is a multiple of D_3. Also, applying to any member of the pencil of conics $A + KB = 0$ the same process as was applied to A would yield a combinative quartic of two line sections whose coefficients are of the same degree as (19); requiring any one of these to be apolar to (10) would yield a combinative cubic expressible in terms of A_3 and B_3.

Cubic on the Flexes of the R^4.

In general no relation of the form

$$aH + bD_3 + cP^3 + dPQ = 0 \tag{27}$$

exists, as may easily be shown by making $b_i = e_i = 0$ in (1) and comparing the resulting values of the letters used in (27). But if we can find a cubic of the system (27) that cuts out *one flex without imposing any condition on the curve, this cubic cuts out all the flexes.* We find only one system of values of a, b, c, and d which satisfy these conditions; hence there is only one *combinant* cubic on the flexes of the R^5. Its equation may be written in either of the following forms,

$$81H - 3{,}125D_3 + 2{,}296P^3 - 2{,}100PQ = 0, \tag{28}$$

or

$$375A_3 - 7{,}875B_3 - 100PQ - 32P^3 = 0. \tag{29}$$

The Flex Tangents of the R^5.

The product of the nine flex tangents of the R^5 can be obtained by finding the condition that there shall be a member of the pencil

$$(\alpha t)^5 + k\,(\beta t)^5 = 0 \tag{30}$$

which contains a cubed factor. This condition is imposed by requiring the vanishing of the matrix

$$(\alpha t)^3\,(\beta t)^3 \begin{vmatrix} \alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2 \\ \beta_1^2 & \beta_1\beta_2 & \beta_2^2 \end{vmatrix}; \tag{31}$$

that is, a value of t must simultaneously satisfy the three sextics

$$(32)\qquad (\alpha t)^3(\beta t)^3\begin{vmatrix}\alpha_1^2 & \alpha_1\alpha_2\\ \beta_1^2 & \beta_1\beta_2\end{vmatrix}=0,\qquad (\alpha t)^3(\beta t)^3\begin{vmatrix}\alpha_1^2 & \alpha_2^2\\ \beta_1^2 & \beta_2^2\end{vmatrix}=0,$$
$$(\alpha t)^3(\beta t)^3\begin{vmatrix}\alpha_1\alpha_2 & \alpha_2^2\\ \beta_1\beta_2 & \beta_2^2\end{vmatrix}=0.$$

If it is understood that the translation scheme explained in the first part of this section has been carried out, these sextics of (32) can be written in the form

$$(33)\qquad \begin{cases}\alpha t^6+3\beta t^5+(3\gamma+6\lambda)t^4+(\delta+8\mu)t^3+(3\varphi+6\chi)t^2+3\mu' t+\lambda'=0,\\ \beta t^6+(3\gamma+3\lambda)t^5+(3\delta+9\mu)t^4+(\psi+9\varphi+8\chi)t^3\\ \qquad\qquad +(3\delta'+9\mu')t^2+(3\gamma'+3\lambda')t+\beta'=0,\\ \lambda t^6+3\mu t^5+(3\varphi+6\chi)t^4+(\delta'+8\mu')t^3+(3\gamma'+6\lambda')t^2+3\beta' t+\alpha'=0.\end{cases}$$

If equations (33) are multiplied by t, and then by t^2, we obtain 6 equations which together with any two of (33) may be solved for $t^8, t^7, t^6, t^5, t^4, t^3, t^2$, and t, and these values substituted in the third equation of (33). Evidently this is the same as the 9-rowed determinant of the 9 simultaneous equations in eight variables obtained in the above way. The method used here for finding the equation of the 9 flex tangents of the R^5 can be used for finding the product of the $3n-6$ flex tangents of the R^n, the general application being so obvious that a formal statement is unnecessary.

The point equation of the R^5 is obtained as the eliminant of (3) and (4). As Salmon gives this explicitly as a five-rowed determinant,* it need not be given here. The work of finding combinants of (3) and (4) could be extended indefinitely, but we believe that those already written out together with invariants and simultaneous invariants† of equations (9), (10), (19) and (20) are a sufficient algebraic basis for the analytic treatment of covariant combinant loci of the R^5 in the plane, which is the same as the invariant theory of the R^5 in space. Hence we shall point out this relation between the combinant curves of the R^5 in the plane and the invariants of the R^5 in space and illustrate the principle by means of several examples.

* SALMON, *Higher Algebra*, Fourth Edition, p. 82.

† Extensive use can be made of the system of combinants of two binary quartics found in SALMON, *Higher Algebra*, Fourth Edition, pp. 219–223.

Invariants of the R^5 in Space.

It is shown in MEYER's *Apolarität und rationale Curven* * that every R^n in d dimensions is symmetrically represented by $n - d$ binary forms of order n; when $n - d = 2$ this means that two binary forms of degree n symmetrically represent a R^n in $n - 2$ dimensions; that is, two binary quartics symmetrically represent an R^4 in the plane; two binary quintics symmetrically represent the R^5 in space. In the plane all line sections of the R^4 are apolar to the pencil of quartics which symmetrically represent the curve; in space any linear combination of the two binary quintics which symmetrically represent the R^5 in space is apolar to all plane sections of this R^5. By the application of Grassmann's theorem it is possible by a simple substitution to obtain the combinants of the two binary quintics associated with the R^5 in space from two other binary quintics; taking the latter two binary quintics as (3) and (4), the combinants of the two former quintics which are invariants of the R^5 in space are obtained in the following manner. Let the two binary quintics (3) and (4) be rewritten in the form

$$\begin{aligned} &at^5 + 5bt^4 + 10ct^3 + 10dt^2 + 5et + f = 0, \\ &a't^5 + 5b't^4 + 10c't^3 + 10d't^2 + 5e't + t' = 0; \end{aligned} \tag{34}$$

let the parametric equations of the R^5 in space be written *without binomial coefficients* in the form

$$x_i = a_i t^5 + b_i t^4 + c_i t^3 + d_i t^2 + e_i t + f_i \quad (i = 0, 1, 2, 3). \tag{35}$$

The matrix of coefficients of (34) is

$$\left\| \begin{matrix} a & b & c & d & e & f \\ a' & b' & c' & d' & e' & f' \end{matrix} \right\|; \tag{36}$$

the matrix of coefficients of (35) is

$$\left\| \begin{matrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{matrix} \right\|. \tag{37}$$

If for any two-rowed determinant of (36) the four-rowed determinant of (37) (obtained by striking out the two columns of (37) which contain the

* See in particular § 3.

letters of the two-rowed determinant of (36)) is substituted, the combinants of (34) become the combinants of the two binary quintics which symmetrically represent the R^5 in space, and are actually invariants of this curve. The scheme of substitutions which must be made to transform the combinants of (3) and (4) into invariants of the R^5 in space whose parametric equations are (35), is obtained from (7) and is

$$(38)\quad \begin{array}{llll} \alpha = |\,cdef\,|, & \beta = |\,bdef\,|, & \gamma = |\,bcef\,|, & \delta = |\,bcdf\,|, \quad \lambda = |\,adef\,|, \\ \alpha' = |\,abcd\,|, & \beta' = |\,abce\,|, & \gamma' = |\,abde\,|, & \delta' = |\,acde\,|, \quad \lambda' = |\,abcf\,|, \\ \mu = |\,acef\,|, & \mu' = |\,abdf\,|, & \psi = |\,bcde\,|, & \varphi = |\,acdf\,|, \quad \chi = |\,abef\,|. \end{array}$$

If these substitutions are made in the combinants of (3) and (4), the transformed expressions are invariants of (35). By the same transformation equations (9), (10), (19) and (20) become binary forms whose roots are sets of covariant points on (35). A very artificial algebraic interpretation could be given to all the combinants of (3) and (4) and to the sets of points represented by (9), (10), (19) and (20), but this would be mere pedantry. It is the purpose of this paper to form an algebraic foundation for analytic work, and geometrical interpretations are a secondary consideration. In order however to illustrate more fully the significance of the work, several applications which appear more or less on the surface are in place.

The combinant P transformed by (38) becomes the condition for the two quintics which symmetrically represent (35) to be apolar; each of necessity is self-apolar; these two quintics are apolar to all plane sections; hence P as an invariant of (35) is the condition for sets of the fundamental involution on (35) to be cut out of the R^5 (35) by planes on a line. The eliminant of (3) and (4) as an invariant of (35) is the condition for the R^5 to have a pentatactic plane or a plane cutting in 5 consecutive parameters. The equation (9) by means of (38) becomes the equation whose roots are tetratactic planes of the R^5. The discriminant of this octavic breaks up into two factors, one of which is the invariant just mentioned; the other factor is the transform of the 9-flex tangents of the R^5 in the plane, and its vanishing conditions the correspondent of the cusp in the plane for the R^5 in space.

The conditions for the R^5 of (35) to have a node may be imposed by making $a_i = f_i$ in (35), for then 0 and ∞ are two different parameters which yield the coördinates of the same point. This condition* makes

$$(39)\quad \begin{cases} \mu = \mu' = \lambda = \lambda' = \varphi = \chi = 0, \\ \alpha = \delta', \quad \alpha' = \delta, \quad \beta = \gamma', \quad \beta' = \gamma, \end{cases}$$

* The identities (8) hold after the transformation (38) is made.

in (38). Let us call the I_3 of equation (20) G and 12 times the I_2 of (20) N. Also, let

$$12K = Q - P^2 \tag{40}$$

and

$$3J = K^2 - N. \tag{41}$$

The conditions for (35) to have a node may be put in the form

$$K^3 - 72KJ + 216\,G = 0; \tag{42}$$

for (42) is satisfied when the conditions (39) hold, but is not true in general.

DARTMOUTH COLLEGE,
June, 1912.